Math 425b, Spring 2021 **Name:** ____________________

Final Exam

5/4–5/7

This exam contains 31 pages (including this cover page) and 7 problems. **Don't be concerned by the number of pages**, but do take a look at Problem 7 which occupies 13 of the pages. **Problem 7 has two options and the first option, which is to prove one of the big theorems that you didn't prove on Midterm 2 (rather than doing the problem outlined on these 13 pages), is probably a good deal shorter and easier**. I included Option 2 in Problem 7 because it's interesting and ties together many of the ideas from the first part of the course, but it may be better to return to it after you've finished the exam.

There are 136 points on this exam in total; as with the midterms, the relevant number for grade calculations is your percentage score ((your score times 100)/136).

One way to take the exam is to print it out, take it by hand as usual, and then scan and upload the result into Gradescope. Other methods are allowed, e.g. you can write on a separate sheet / sheets of paper or a tablet, or use LaTeX as long as you adhere to the time limits. In any case, make sure your submission includes your name to avoid any Gradescope issues.

The time limit for the exam is: 2 time blocks (first block 4 hours, second block 5 hours), with a break of any length in between (e.g. if you have questions about any of the problems, you can email me and wait to take your second time window until I reply, but please make sure you're completely done with the exam by Friday night). You are not allowed to consult any course- or exam-related resources during the break (books, notes, etc.).

You are required to write in complete sentences on this exam, with the usual exceptions for abbreviations, symbols, etc. permitted as on the homework.

You may *not* use books, notes, calculator, the Internet, or other outside resources on this exam. You are expected to conduct yourself with academic integrity in all aspects of this exam; please let me know if any concerns arise.

1. (18 points) (a) (6 points) Let X be a set and let (Y, d) be a metric space. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from X to Y. Define what it means for $(f_n)_{n=1}^{\infty}$ to converge uniformly to some function $f : X \to Y$. Also, given a sequence of functions $(g_n)_{n=1}^{\infty}$ from X to a normed vector space $(V, \|\cdot\|)$, define what it means for the series

$$\sum_{n=1}^{\infty} g_n$$

to converge uniformly.

(b) (6 points) Recall that one way to characterize convergence of doubly-infinite series $\sum_{n=-\infty}^{\infty} g_n$ is by requiring that $\sum_{n=0}^{\infty} g_n$ converges and that $\sum_{n=1}^{\infty} g_{-n}$ converges (pointwise, uniformly, etc.). Show that the doubly infinite series

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$$

converges uniformly for $t \in [a, b]$, assuming that $0 < a < b$.

Hint: One approach is to use the Weierstrass M-test; can you bound the terms of the series above by constants that don't depend on t, such that these constants have finite sum? If you have a sum with an n^2 and you'd know it converged if it was n instead of n^2, the comparison test might be useful (choose the "smaller" series to be zero in term n when n is not a perfect square).

(More space for previous problem)

(c) (6 points) For a fixed $t > 0$, consider the function $g_t(x) = e^{-\pi t x^2}$ (a Gaussian with some normalization). **You may assume without proof** that the Fourier transform $\hat{g}_t(\xi)$ exists and is equal to $\frac{1}{\sqrt{t}} e^{\pi \xi^2 / t}$; this is a standard result. Show that

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 / t}$$

(note that convergence of both sums follows from the previous problem, since uniform convergence implies pointwise convergence).

Hint: Apply the Poisson summation formula! If you forget this formula, you might be able to guess it from the above equality.

(More space for previous problem, although if you find yourself doing long computations then you may be using the wrong formula).

2. (18 points) (a) (8 points) Let V and W be normed vector spaces and let $T : V \to W$ be a linear transformation. Define the operator norm $\|T\|_{\text{op}}$ of T (it might be finite or infinite; there are multiple equivalent answers you could give for this definition and they're all okay).

(b) (10 points) Let V, W, Z be normed vector spaces and let $S : V \to W$, $T : W \to Z$ be linear transformations; assume that $\|T\|_{\text{op}}$ and $\|S\|_{\text{op}}$ are finite. Show that

$$\|T \circ S\|_{\text{op}} \leq \|T\|_{\text{op}} \|S\|_{\text{op}}.$$

Hint: Start with the definition of $\|T \circ S\|_{\text{op}}$ as the supremum over certain vectors v of a ratio of certain norms (this may also serve as a hint for the first part of this problem). Multiply the numerator and denominator of this ratio by $\|S(v)\|$ where v is the vector you're taking the supremum over; you should argue separately that you don't need to worry about vectors with $S(v) = 0$. Then apply properties of suprema to derive the inequality you want.

(More space for previous problem)

3. (18 points) (a) (9 points) Let U be an open subset of $\mathbb{R}^n$ and let F be a function from U to $\mathbb{R}^m$. Define what it means for F to be differentiable at a point $p \in U$ with total derivative given by a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

(b) (9 points) **Choose one** of the following two problems:

- Show that if U is an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^m$ is differentiable at p, then F is continuous at p.

  Hint: You want to show that $\lim_{v \to 0} F(p + v) = F(p)$; to do this, you can use the "approximation formula" for $F(p + v)$ in terms of its total derivative. Try to show that this formula approaches $F(p)$ as $v \to 0$; to analyze the remainder term $R(v)$ in the formula, the definition of differentiability is useful.

- Show, directly from the definition of differentiability, that if the total derivative of F at p exists then it is unique (i.e. if F is differentiable at p with total derivative T, and the same is true with T replaced by T', then $T = T'$.

  Hint: Show that if both T and T' work in the definition of differentiability of F at p, then $\lim_{v \to 0} \frac{T(v) - T'(v)}{\|v\|} = 0$. Assuming we have $T(v_0) \neq T'(v_0)$ for some vector v_0, try to derive a contradiction by scaling v_0.

(More space for previous problem)

4. (25 points) Prove **one** of the following theorems (your choice). If you prove more than one, only the first will be graded. You may assume any lemmas that you need; only the main proof of the theorem is required. Extended hints for each can be found below.

- Multivariable chain rule: if
  - $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open,
  - $p \in U$,
  - $F : U \to V$ and $G : V \to \mathbb{R}^k$ are functions,
  - F is differentiable at p with total derivative A, and
  - G is differentiable at $F(p)$ with total derivative B,

  then $G \circ F$ is differentiable at p with total derivative BA (it's useful to view A and B as matrices here, corresponding to linear transformations by using the standard bases for $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^k$).
- C^1 functions are differentiable: if U is an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^m$ is a C^1 function on U, then F is differentiable at each point $p \in U$ with total derivative at p having standard-basis matrix

$$(Jf)_p = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}.$$

- Inverse function theorem: if
  - $U_0 \subset \mathbb{R}^n$ is open with $p \in U_0$,
  - $f : U_0 \to \mathbb{R}^m$ is a C^r function for some $1 \leq r \leq \infty$, and
  - $(Df)_p$ is invertible,

  then $m = n$ and there exist open subsets $U \subset U_0$, $V \subset \mathbb{R}^n$, containing $p, f(p)$ respectively, such that f is a C^r diffeomorphism from U to V.

Hint for chain rule: Let $q = F(p)$. Let $R_F(v) = F(p+v) - F(p) - Av$ and $R_G(w) = G(q+w) - G(q) - Bw$. Similarly, let $R_{G\circ F}(v) = G(F(p+v)) - G(F(p)) - BAv$; you want to show that $\frac{R_{G\circ F}(v)}{\|v\|} \to 0$ as $v \to 0$.

In the formula for $R_{G\circ F}(v)$, start by writing out $F(p+v)$ in terms of its linear approximation and remainder term $R_F(v)$. The result should be $q = F(p)$ plus something else; call this other term w and write out $G(q+w)$ in terms of its linear approximation and remainder term $R_G(w)$. You should get some cancellation in your overall expression for $R_{G\circ F}(v)$; show that

$$R_{G\circ F}(v) = BR_F(v) + R_G(Av + R_F(v)).$$

It suffices to show each of these terms, divided by $\|v\|$, goes to zero as $v \to 0$. Deal with the first term first, then consider the more complicated second term. Show that for v with $Av + R_F(v) = 0$, the second term always vanishes, so we can restrict attention to v with $Av + R_F(v) \neq 0$.

In this case, multiply the numerator and denominator of $\frac{R_G(Av+R_F(v))}{\|v\|}$ by the same quantity (chosen so that you can apply differentiability of G at q). You will get a product of two fractions, one of which you can show approaches zero as $v \to 0$, and the other of which you can show is bounded as $v \to 0$.

Hint for C^1 implies differentiability: Let $J = (JF)_p$ and let $R(v) = F(p+v) - F(p) - Jv$; you want to show $\frac{R(v)}{\|v\|} \to 0$ as $v \to 0$. For $1 \le i \le m$, let $R_i(v)$ be the i^{th} coordinate of $R(v)$; write out $R_i(v)$ using explicit partial derivatives for the term coming from the i^{th} row of Jv. It suffices to show that for each fixed i, we have $\frac{R_i(v)}{\|v\|} \to 0$ as $v \to 0$.

To do this, given $\varepsilon > 0$, choose $\delta > 0$ such that if $\|v\| < \delta$ then $p + v \in U$ and

$$\left| \frac{\partial F_i}{\partial x_j}(p+v) - \frac{\partial F_i}{\partial x_j}(p) \right| < \varepsilon/n$$

for $1 \le j \le n$ (why is it possible to choose δ like this?). Your goal will be to show that if $\|v\| < \delta$ then $\frac{|R_i(v)|}{\|v\|} < \varepsilon$, proving the theorem.

To prove this inequality, write out $R_i(v)$ even more explicitly, expanding out any matrix multiplications that appeared in your above formula. You should have $F_i(p+v) - F_i(p)$ minus a sum of n terms; the strategy will be to write $F_i(p+v) - F_i(p)$ similarly as a (telescoping) sum of n terms. You will then have n pairs of terms, and you will want to show that each pair has absolute difference at most ε/n.

For the telescoping sum, let e_j be the j^{th} standard basis vector of $\mathbb{R}^n$ for $1 \le j \le n$. It helps to write

$$p + v = p + v_1 e_1 + \cdots + v_n e_n;$$

from here, look for a natural way to write $F_i(p+v) - F_i(p)$ as a telescoping sum

$$F_i(p_n) - F_i(p_{n-1}) + F_i(p_{n-1}) - \cdots - F_i(p_1) + F_i(p_1) - F_i(p_0)$$

for some points $p_0, \ldots, p_n$ with $p_0 = p$ and $p_n = p + v$.

You want to set things up above so that $F_i(p_j) - F_i(p_{j-1})$ can be handled using the single-variable mean value theorem, so p_j and p_{j-1} should differ only in their j^{th} coordinates. Once you've chosen the right points p_j, define a straight-line path σ_j from p_{j-1} to p_j, in such a way that the existence of the partial derivative $\frac{\partial F_i}{\partial x_j}$ implies the differentiability of $F_i \circ \sigma_j$ (and such that the derivative of $F_i \circ \sigma_j$ at t is $\frac{\partial F_i}{\partial x_j}$ evaluated at $\sigma_j(t)$).

From here, the single-variable mean value theorem should allow you to rewrite $F_i(p_j) - F_i(p_{j-1})$ in a more useful way. Returning to the goal, use this rewriting together with the assumption $\|v\| < \delta$ to show that $\frac{R_i(v)}{\|v\|} < \varepsilon$ as desired.

Hint for inverse function theorem: To apply the implicit function theorem, define $F : U_0 \times \mathbb{R}^n \to \mathbb{R}^n$ by $F(x, y) = f(x) - y$ (note that $U_0 \times \mathbb{R}^n$ is an open subset of $\mathbb{R}^{2n}$). Let $q = f(p)$ and compute the Jacobian matrix of F at (p, q) as a block matrix (there should be a left block and a right block, both of size $n \times n$, and both invertible).

Use invertibility of the left block (the less trivial one) and apply the implicit function theorem: you should get open neighborhoods $U_p \subset U_0$ and $V_q \subset \mathbb{R}^n$ of p and q respectively, and a unique function $h : V_q \to U_p$ such that

$$F^{-1}(0) \cap (U_p \times V_q) = \mathrm{graph}(h)$$

(you also know that h is a C^r function). Use this property to deduce that

$$f \circ h = \mathrm{id}_{V_q}\,.$$

Now run the same argument in reverse. Define $G : U_q \times V_p \to \mathbb{R}^n$ by $G(x, y) = x - h(y)$. Compute the Jacobian matrix of G at (p, q) as a block matrix; show the right block is invertible and use the implicit function theorem to get an open neighborhood $U'_p \times V'_q \subset U_p \times V_q$ of (p, q) and a unique function $g : U'_p \to V'_q$ such that

$$G^{-1}(0) \cap (U'_p \times V'_q) = \mathrm{graph}(g).$$

Use this property to deduce that

$$h \circ g = \mathrm{id}_{U'_p}\,.$$

Now clean things up to get an actual local inverse map for f: first show that for points x in U'_p, we have $g(x) = f(x)$ (i.e. that g is the restriction of f to U'_p). This should be short; start by writing $g(x) = f(h(g(x))$ (why can you do this?).

In a final "cleaning" step, let $U := U'_p$ and $V := h^{-1}(U)$. Show that the conclusions of the inverse function theorem are satisfied: f maps U into V, h maps V into U, and the compositions $h \circ f$ and $f \circ h$ are id_U and id_V respectively.

(more space for previous problem)

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5. (16 points) Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(x, y, z) = (4x^2 + 4y^2 - 4z^2,\ x^2 + y^2 + z^2).$$

(a) (8 points) Compute the Jacobian matrix of F, a 2×3 matrix whose entries should depend on (x, y, z).

(b) (8 points) Consider the level set $F^{-1}(1, 1) \subset \mathbb{R}^3$ (this level set is the intersection of a hyperboloid and a sphere in $\mathbb{R}^3$). Show that for all points (x, y, z) in the level set $F^{-1}(1, 1)$, the Jacobian matrix of F has maximal rank (so that the hypotheses of the implicit function theorem hold).

Hint: At a point (x, y, z), if you can select two of the three columns of $(Jf)_{(x,y,z)}$ to get a matrix with nonzero determinant, then $(Jf)_{(x,y,z)}$ has maximal rank. Show that if the determinant is zero when you take columns 1 and 3, and the determinant is also zero when you take columns 2 and 3, then it's not possible to have $4x^2 + 4y^2 - 4z^2 = 1$ and $x^2 + y^2 + z^2 = 1$.

(More space for previous question)

6. (16 points) (a) (8 points) Let $\beta = (x^2y + z)dx + (xyz)dy + (x + yz^2)dz$, a differential 1-form on $\mathbb{R}^3$. Compute $d\beta$.

(b) (8 points) Let $P(x)$ and $Q(x)$ be smooth functions from $\mathbb{R}$ to $\mathbb{R}$, and consider the differential 1-form

$$\alpha = (P(x)y - Q(x))dx + dy$$

on $\mathbb{R}^2$. Let

$$\mu(x) = e^{\int_0^x P(t)dt}.$$

Prove that the differential 1-form

$$\mu\alpha = \mu(x)(P(x)y - Q(x))dx + \mu(x)dy$$

satisfies $d(\mu\alpha) = 0$.
Remark. A function μ as above is often called an *integrating factor* for α.

(More space for previous problem)

7. (25 points) For this question, you have a choice:

- **Option 1:** look back at midterm 2. Out of { Arzelá–Ascoli theorem, Weierstrass approximation theorem, Stone–Weierstrass theorem, Picard–Lindelöf theorem }, pick one that you didn't prove on midterm 2 and prove it (you can use the hints from midterm 2), or:
- **Option 2:** do the following problem, exploring the functional equation for the Riemann ζ function (this is sort of like a homework problem with some new material along with hints).

Note: if you do attempt the following problem, which I think is harder and more time-consuming than the proofs from midterm 2, I recommend doing the other problems on the exam first so you don't run into time issues. It might be a safer bet to do Option 1 and then return to this problem once you're done with the exam if you're interested.

(a) (5 points) Let $c > 0$. For any complex number s, prove that the improper Riemann integral

$$\int_1^\infty t^{s-1}e^{-ct}dt$$

converges (**see the hint** for a result you can assume without proof). If the real part of s is greater than 0, prove that the improper Riemann integral

$$\int_0^\infty t^{s-1}e^{-ct}dt$$

converges (when $c = 1$ this integral defines the Γ function $\Gamma(s)$).

Hint: For the first part, it suffices to show that $\int_1^b t^{s-1}e^{-ct}dt$ is Cauchy in b, i.e. that $\left|\int_{b_1}^{b_2} t^{s-1}e^{-ct}dt\right|$ is small for large b_1, b_2. To show this, pick b_1, b_2 large enough so that for the t-interval under consideration, $|t^{s-1}| \leq e^{ct/2}$ (note that $|t^s| = t^{\mathrm{Re}(s)}$ for positive t). You know it's possible to choose these b_1, b_2 since $\lim_{t\to\infty} \frac{t^K}{e^{ct/2}} = 0$ for any K; **you may assume this result without proof** although it follows from L'Hopital's rule once you assume (WLOG) that K is an integer. Now use convergence of the improper Riemann integral

$$\int_1^\infty e^{-ct/2}dt$$

which you can show by a direct computation.

For the second part, the first part shows you only need to consider $\int_0^1 t^{s-1}e^{-ct}dt$; you can argue that $\int_a^1 t^{s-1}e^{-ct}dt$ is Cauchy in a by estimating away the e^{-ct} factor (it's ≤ 1) and using that the real part of s is greater than 0.

(More space for previous problem)

(b) (5 points) For any complex number s with real part > 1, we want to define (a variant of) the Riemann ξ function using the improper Riemann integral formula

$$\xi(s) := \int_0^\infty t^{\frac{s}{2}-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt.$$

Show that this improper Riemann integral exists and is given by the formula

$$\xi(s) = 2\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

where

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt$$

is the Γ function considered above and

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}$$

is the Riemann ζ function (this series converges when $\mathrm{Re}(s) > 1$ by the results we covered on p-series). **You may assume the following result without proof**, which follows from HW 4:

Proposition. *Let $(A, \preceq)$ be a directed set and consider a sequence $f_n(a)$ of real-valued nets $f_n : A \to \mathbb{R}$. Assume that:*

- *For each fixed n, the net $f_n(a)$ converges (in a) to a limit $L_n \in \mathbb{R}$.*
- *The nets f_n converge uniformly (as a sequence of functions $A \to \mathbb{R}$) to some net $f : A \to \mathbb{R}$.*

Then the limits L_n converge to some $L \in \mathbb{R}$, and $\lim_a f(a)$ exists and equals L; in other words

$$\lim_a \lim_{n\to\infty} f_n(a) = \lim_{n\to\infty}\lim_a f_n(a).$$

Hint: The strategy for the hint will be to first look at the integral

$$\int_a^b t^{\frac{s}{2}-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt$$

for finite $0 < a < b$, then try to consider the limit as $a \to 0, b \to \infty$. For the integral with finite limits (a, b), you're integrating with respect to dt and you know from the first problem on the exam that $\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$ converges uniformly for $t \in [a, b]$. Conclude that $\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$ is a continuous (and thus Riemann integrable) function of $t \in [a, b]$; the same is true for the entire integrand above, so that the finite integral from a to b makes sense.

Next, use what you know about how Riemann integrals interact with uniformly convergent series of functions to bring the infinite sum outside the integral sign. It might be convenient at this point to note that the "minus one" in the parenthesized part of the integrand just cancels the $n = 0$ term of the doubly infinite sum, which is then symmetric, so that the parenthesized part is the same as

$$2\sum_{n=1}^{\infty} e^{-\pi t n^2}.$$

You should get a formula for the finite integral from a to b that looks like

$$2\sum_{n=1}^{\infty}\int_a^b \dots dt$$

(the integrand inside the sum should appear related to the integrand of the Γ function, although not quite identical).

Now try to take the limit as $a \to 0$ and $b \to \infty$, which you want to show can be interchanged with $\sum_{n=1}^{\infty}$. The overall plan is to use the above proposition on uniform convergence and limits of nets, with A the set of pairs (a, b) of real numbers with $0 < a < b$ and $(a, b) \preceq (a', b')$ if $a' < a$ and $b' > b$. Adapting the notation of the above proposition, the net $f_n(a, b)$ should be a finite integral $\int_a^b$ of the n^{th} partial sum $\sum_{m=1}^{n}$ of the series (equivalently, you could take the integral inside the partial sum).

- To show $\lim_{a\to 0, b\to\infty} f_n(a, b)$ exists for each fixed n, you can use part (a) of this problem.
- To show the series $\sum_{n=1}^{\infty}\int_a^b \dots dt$ converges uniformly in the variables a, b (ranging over all pairs (a, b) with $0 < a < b$), you can show it's uniformly Cauchy: for large enough N, M, you want to show that the sum from N to M of $\int_a^b \dots dt$ is uniformly small independently of a and b (even for very small a and very large b). For each n, you can use the u-substitution $u = \pi n^2 t$ (limits of integration will be from $\pi n^2 a$ to $\pi n^2 b$), then show the absolute value of the result is less than or equal to some coefficient (constant in n, involving $\Gamma(s/2)$ and another factor) times $\frac{1}{n^s}$. Finally, use convergence of the ζ function for s with real part > 1 to show that $\sum_{n=N}^{M}\frac{1}{n^s}$ is small for large N, M.

It should follow that

$$\lim_{a\to 0, b\to\infty}\lim_{n\to\infty} f_n(a, b) = \lim_{n\to\infty}\lim_{a\to 0, b\to\infty} f_n(a, b)$$

(in particular, the limit on the left-hand side exists). You can evaluate the right-hand side using the same u-substitution as above, and you should get the formula you're trying to prove (relating $\xi(s)$ to $\Gamma(\frac{s}{2})$ and $\zeta(s)$).

(More space for previous problem)

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(c) (5 points) Show that the improper Riemann integral integral

$$\int_1^\infty t^{-1}\left(t^{s/2}+t^{(1-s)/2}\right)\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}-1\right)dt$$

makes sense for all complex s.

Hint: Note that the lower limit is $t=1$ rather than $t=0$ (this makes things simpler). As in the previous part, you can start by looking at the integrals $\int_1^b$ for finite b; you want to show their limit exists as $b\to\infty$. You can apply the above proposition on uniform convergence and limits of nets again:

- For each fixed n, you can use part (a) of the problem to show that $\lim_{b\to\infty} f_n(b)$ exists, with $f_n(b)$ defined as

$$f_n(b)=2\int_1^b t^{-1}\left(t^{s/2}+t^{(1-s)/2}\right)\left(\sum_{m=1}^{n} e^{-\pi t m^2}\right)dt$$

  (it may help to note that for large enough constants A, K, one has $t^{-1}\left(t^{s/2}+t^{(1-s)/2}\right)\le At^K$ for all $t\ge 1$).

- To show $f_n(b)$ converges as $n\to\infty$, uniformly in $b\in[1,\infty)$, it suffices to show that for large enough N, M, the quantity

$$2\sum_{m=N}^{M}\int_1^b t^{-1}\left(t^{s/2}+t^{(1-s)/2}\right)e^{-\pi t m^2}dt$$

  is small independently of b. Write

$$e^{-\pi t m^2}=e^{-\pi t m^2/2}e^{-\pi t m^2/2}\le e^{-\pi m^2/2}e^{-\pi t/2},$$

  pull the factor $e^{-\pi m^2/2}$ outside the integral, and show that the remaining integral is $\le C$ for some constant C independent of both b and m. You'll be reduced to showing that

$$\sum_{m=N}^{M} e^{-\pi m^2/2}$$

  is small for large N, M.

It should follow that the improper Riemann integral in the problem statement exists for all complex s (it equals

$$2\sum_{n=1}^{\infty}\int_1^\infty t^{-1}\left(t^{s/2}+t^{(1-s)/2}\right)e^{-\pi t n^2}dt,$$

which also exists).

(More space for previous problem)

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(d) (5 points) Show that for complex s with $\mathrm{Re}(s) > 1$, we have

$$\xi(s) = -\frac{2}{s(1-s)} + \int_1^\infty t^{-1}\left(t^{s/2} + t^{(1-s)/2}\right)\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt.$$

Hint: Split the original integral definition of $\xi(s)$ into integrals from 0 to 1 and 1 to ∞:

$$\xi(s) = \int_0^1 t^{\frac{s}{2}-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt + \int_1^\infty t^{\frac{s}{2}-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt.$$

For the integral from 0 to 1, do the u-substitution $u = 1/t$ so you get an integral from $u = 1$ to $u = \infty$. *Using the first problem on this exam*, show that

$$\xi(s) = \int_1^\infty t^{\frac{-s-1}{2}}\left(\sum_{n=-\infty}^{\infty} e^{\pi t n^2} - \frac{1}{\sqrt{t}}\right) du + \int_1^\infty t^{\frac{s}{2}-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt$$

(this is the key step). The difference between this expression and the integral in the statement of the problem is

$$\int_1^\infty t^{\frac{-s-1}{2}}\left(1 - \frac{1}{\sqrt{t}}\right) dt;$$

show this integral evaluates to $\frac{-2}{s(1-s)}$.

(more space for previous problem)

(e) (5 points) Note that the previous part of the problem defines $\xi(s)$ for all complex $s \neq 0, 1$. Since we have

$$\zeta(s) = \frac{\pi^{s/2}\xi(s)}{2\Gamma(s/2)}$$

for $\mathrm{Re}(s) > 1$ and $\Gamma(s)$ is defined for $\mathrm{Re}(s) > 0$, the above formula extends the domain of definition of $\zeta(s)$ to all s with $\mathrm{Re}(s) > 0$. In fact, the Γ function satisfies the following property which you may **assume without proof**:

Proposition. *For* $0 < \mathrm{Re}(s) < 1$*, we have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Thus, for $\mathrm{Re}(s) < 1$ we can define $\Gamma(s) := \frac{\pi}{\sin(\pi x)\Gamma(1-s)}$ and this agrees with our original definition on $0 < \mathrm{Re}(s) < 1$. It follows that the Γ function (and thus by above the ζ function) extends to the whole complex plane (minus some "poles"), and that the above equation relating the Γ and sine functions is valid on the complex plane.

You may also **assume the following property without proof**, due to Legendre:

Proposition. *For* s *in the complex plane, we have*

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s).$$

Show that Riemann's functional equation for the ζ function,

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),$$

is satisfied.

Hint: Start by noting that since the definition of $\xi(s)$ treats s and $1-s$ symmetrically, we have $\xi(s) = \xi(1-s)$. Expand this equation out using

$$\xi(s) = 2\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

on both sides, and rearrange so the equation reads $\zeta(s) = \ldots\zeta(1-s)$. You want to show that the expression "…" you get here agrees with $2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)$.

To do this, first use a proposition above to express $\sin\left(\frac{\pi s}{2}\right)$ in terms of gamma functions. In the equation you're trying to prove, there should be a $\Gamma(s/2)$ factor that cancels on each side, simplifying things. Once you've made this simplification, you can apply the other proposition above.

(More space for previous problem)

(f) (0 points) Given the above extension of the function $\zeta(s)$ to the complex plane, show that $\zeta(s) \neq 0$ unless $s = -2, -4, -6, \ldots$, or $\mathrm{Re}(s) = 1/2$. **No points but you get everlasting fame and a million dollars.**