MATH 425b Final Exam

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Problem 1

(a) Let X be a set and (Y,d) a metric space. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions $X \to Y$. Define what it means for $\{f_n\}_{n=1}^\infty$ to converge uniformly to some function $f : X \to Y$. Also, given a sequence of functions $\{g_n\}_{n=1}^\infty$ from X to a formed vector space $(V, \|\cdot\|)$, define what it means for the series $\sum_{n=1}^\infty g_n$ to converge uniformly.

(b) Recall that one way to characterize convergence of doubly-infinite series $\sum_{n=-\infty}^\infty g_n$ is by requiring that $\sum_{n=0}^\infty g_n$ converges and that $\sum_{n=1}^\infty g_{-n}$ converges (pointwise, uniformly, etc.). Show that the doubly infinite series

$$\sum_{n=-\infty}^\infty e^{-\pi t n^2}$$

converges uniformly for $t \in [a,b]$ assuming $0 < a < b$.

(c) For a fixed $t > 0$, consider the function $g_t(x) = e^{-\pi t x^2}$, a Gaussian with some normalization. You may assume without proof that the Fourier transform $\hat{g}_t(\xi)$ exists and is equal to $e^{\pi\xi^2/t}/\sqrt{t}$. Show that

$$\sum_{n=-\infty}^\infty e^{-\pi t n^2} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^\infty e^{-\pi n^2/t}.$$

[Hint: Poisson summation formula.]

Solution.

(a) $\{f_n\}_{n=1}^\infty$ converges uniformly to f if the following criterion is met:

For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n \geqslant N$, $d(f_n(x), f(x)) < \epsilon$ for <u>all</u> x.

Likewise, $\sum_{n=1}^\infty g_n$ converges uniformly if the sequence of its partial sums converges uniformly, i.e., if

$$\{h_k\}_{k=1}^\infty \text{ defined by } h_k := \sum_{i=1}^k g_n$$

converges uniformly.

(b) *Proof.* Notice that if $f(x) = e^{-\pi x n^2}$ for $x \in [a,b]$ as defined in the problem, then $\|f\|_\infty = f(a)$ (because this exponential function is strictly decreasing).

Consider the series of constants $\sum_{n=1}^{\infty} e^{-\pi a n^2}$. Since we only focus on $t \in [a,b]$, each term $e^{-\pi t n^2}$ is bounded by $e^{-\pi a n^2}$. Therefore $\sum_{n=1}^{\infty} e^{-\pi t n^2} \leqslant \sum_{n=1}^{\infty} e^{-\pi a n^2}$. Also, since exponentials are always positive, if we try to bound the series, it does not hurt to add some extra terms:

$$\sum_{n=0}^{\infty} e^{-\pi t n^2} \leqslant \sum_{n=0}^{\infty} e^{-\pi a n^2} = \sum_{k \text{ square}} e^{-\pi a k} < \sum_{k=0}^{\infty} e^{-\pi a k}.$$

The last one is a geometric series with exponential growth rate $1/e < 1$ so it converges to a finite number. Therefore so is the first one, and clearly $\sum_{i=1}^{\infty} e^{-\pi t(-n)^2} = \sum_{i=1}^{\infty} e^{-\pi t n^2} < \sum_{k=1}^{\infty} e^{-\pi a k}$.

We just bounded both singly-infinite series of functions (with respect to $\|\cdot\|_\infty$) by a convergent series of constants and so we can invoke the Weierstraß M-test and conclude that $\sum_{n=0}^{\infty} e^{-\pi t n^2}$ and $\sum_{n=1}^{\infty} e^{-\pi t(-n)^2}$ converges uniformly on $[a,b]$. Hence $\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$ also converges uniformly. □

(c) Poisson summation formula states that $\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$, and this is precisely what the equation is:

$$\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \sum_{\xi=-\infty}^{\infty} \frac{\exp(-\pi \xi^2/t)}{\sqrt{t}} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t}.$$ □

Problem 2

(a) Let V, W be normed vector spaces and let $T : V \to W$ be a linear transformation. Define the operator norm $\|T\|_{\text{op}}$ of T.

(b) Let V, W, Z be normed vector spaces and let $S : V \to W$ and $T : W \to Z$ be linear transformations. Assume that both $\|T\|_{\text{op}}$ and $\|S\|_{\text{op}}$ are finite. Show that

$$\|T \circ S\|_{\text{op}} \leqslant \|T\|_{\text{op}} \|S\|_{\text{op}}.$$

Solution.

(a) $\|T\|_{\text{op}} = \inf\{L > 0 : \|T(v)\| \leqslant L(v) \text{ for all } v \in V\} = \sup_{\|v\|=1} \|T(v)\| = \sup_{\|v\|\leqslant 1} \|T(v)\| = \sup_{v \neq 0} \frac{\|T(v)\|}{\|v\|}$.

(b) *Proof.* The claim is trivial when one of them has 0 operator norm. For the nontrivial case, we will need to use the fact that supremum of product $\leqslant$ product of supremum:

$$\begin{aligned}\|T \circ S\|_{\text{op}} = \sup_{\|v\|\neq 0} \frac{\|TS(v)\|}{\|v\|} &= \sup_{\substack{\|v\|\neq 0 \\ S(v)\neq 0}} \frac{\|TS(v)\|}{\|v\|} = \sup_{\substack{\|v\|\neq 0 \\ S(v)\neq 0}} \frac{\|TS(v)\|}{\|S(v)\|} \cdot \frac{\|S(v)\|}{\|v\|} \\ &\leqslant \sup_{\substack{\|v\|\neq 0 \\ S(v)\neq 0}} \frac{\|TS(v)\|}{\|S(v)\|} \cdot \sup_{\substack{\|v\|\neq 0 \\ S(v)\neq 0}} \frac{\|S(v)\|}{\|v\|} = \sup_{S(v)\neq 0} \frac{\|TS(v)\|}{\|S(v)\|} \|S\|_{\text{op}} \leqslant \|T\|_{\text{op}} \|S\|_{\text{op}}.\end{aligned}$$ □

Problem 3

(a) Let U be an open subset of $\mathbb{R}^n$ and let F be a function from U to $\mathbb{R}^m$. Define what it means for F to be differentiable at $p \in U$ with total derivative given by a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

(b) (Second option) show that if such T exists then it is unique.

Solution.

(a) We say F is differentiable at p with total derivative T if

$$\lim_{v \to 0} \frac{F(p+v) - F(p) - T(v)}{\|v\|} = 0.$$

(b) *Proof.* If T and S are distinct total derivatives of F at p, then substituting S and T into the definition above and subtracting yields

$$\lim_{v \to 0} \frac{S(v) - T(v)}{\|v\|} = \lim_{v \to 0} \frac{(S-T)(v)}{\|v\|} = 0.$$

Now suppose for contradiction that $S - T$ is not the zero linear transformation, i.e., there exists v_0 satisfying $(S-T)(v_0) \neq 0$. We consider cv_0 where $c \in \mathbb{R}^+$. Letting $c \to 0$, we have $cv_0 \to 0$, and so

$$\lim_{c \to 0} \frac{(S-T)(cv_0)}{\|cv_0\|} = \lim_{c \to 0} \frac{(S-T)(v_0)}{\|v_0\|} \neq 0,$$

contradiction. Hence $S - T$ must be the zero transformation, i.e., $S = T$. Hence the uniqueness. $\square$

Problem 4

(Second option) prove that C^1 functions are differentiable: if $U \subset \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^m$ a C^1 function on U, then F is differentiable at all $p \in U$ with total derivative at p having the standard-basis matrix

$$(\mathcal{J}F)_p = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(p) & \cdots & \frac{\partial F_1}{\partial x_n}(p) \\ \vdots & \ddots & \cdots \\ \frac{\partial F_m}{\partial x_1}(p) & \cdots & \frac{\partial F_m}{\partial x_n}(p) \end{bmatrix}.$$

Proof. Let $\mathcal{J} := (\mathcal{J}F)_p$ and let $R(v) := F(p+v) - F(p) - \mathcal{J}v$. We want to show that $R(v)/\|v\| \to 0$ as $v \to 0$. For $1 \leqslant i \leqslant m$, let $R_i(v)$ be the i^{th} coordinate of $R(v)$ and likewise for $F_i(v)$. By definition

$$R_i(v) = F_i(p+v) - F_i(p) - \begin{bmatrix} \frac{\partial F_i}{\partial x_1}(p) & \cdots & \frac{\partial F_i}{\partial x_n}(p) \end{bmatrix} v.$$

It suffices to show that for each i, $R_i(v)/\|v\| \to 0$ as $v \to 0$.

Now let $\epsilon > 0$ be given. We choose $\delta > 0$ satisfying the following:

(1) If $\|v\| < \delta$ then $p + v \in U$ (possible because U is open) and

(2) $\left| \frac{\partial F_i}{\partial x_j}(p+v) - \frac{\partial F_i}{\partial x_j}(p) \right| < \frac{\epsilon}{n}$ for $1 \leqslant j \leqslant n$ (possible because $\frac{\partial F_i}{\partial x_j}$ is continuous).

We claim that this δ satisfies the $\epsilon - \delta$ condition, i.e., if $\|v\| < \delta$ then $|R(v)|/\|v\| < \epsilon$.

To see this, we first rewrite $v = \sum_{j=1}^{n} v_j e_j$ (where e_j's are the standard basis of $\mathbb{R}^n$) and rewrite $R_i(v)$ in the form of a telescoping sum:

$$\begin{aligned} R_i(v) &= F_i(p + \sum_{i=1}^{n} v_j e_j) - F_i(p) - \sum_{j=1}^{n} v_j \frac{\partial F_i}{\partial x_j}(p) \\ &= F_i(p + \sum_{j=1}^{n} v_j e_j) - F_i(p + \sum_{j=1}^{n-1} v_j e_j) - v_n \frac{\partial F_i}{\partial x_n}(p) \\ &\quad + F_i(p + \sum_{j=1}^{n-1} v_j e_j) - F_i(p + \sum_{j=1}^{n-2} v_j e_j) - v_{n-1} \frac{\partial F_i}{\partial v_{n-1}}(p) \\ &\quad + \cdots + F_i(p + v_1 e_1) - F_i(p) - v_1 \frac{\partial F_i}{\partial v_1}(p). \end{aligned}$$

Clearly there are n lines in total, and we will show that each line $< \epsilon/n$ using the MVT. For the $(n+1-k)^{\text{th}}$ line (k^{th} counting from bottom), since

$$g : t \mapsto F_i(p + \sum_{j=1}^{k-1} v_j e_j + t e_k)$$

is differentiable on $[0, v_k]$ (because $\partial F/\partial x_j$ exists on U), it's well-defined to compute its derivative

$$g'(t) = \frac{\partial F_i}{\partial x_k}(p + \sum_{j=1}^{k-1} v_j e_j + t e_k).$$

Therefore $g(v_k) - g(0) = g'(\theta)(v_k - 0)$ for some $\theta \in [0, v_k]$, i.e.,

$$F_i(p + \sum_{j=1}^{k} v_j e_j) - F_i(p + \sum_{j=1}^{k-1} v_j e_j) = v_k \frac{\partial F_i}{\partial x_k}(p + \sum_{j=1}^{k-1} v_j e_j + \theta e_k).$$

Finally, since $\|(v_1, ..., v_{k-1}, \theta, 0, ...)\| \leqslant \|(v_1, ..., v_n)\| = \|v\|$, we have

$$\begin{aligned} \|(n+1-k)^{\text{th}} \text{ line}\| &= \left\| v_k \frac{\partial F_i}{\partial x_k}(p + \sum_{j=1}^{k-1} v_j e_j + \theta e_k) - v_k \frac{\partial F_i}{\partial x_k}(p) \right\| \\ &= |v_k| \left\| \frac{\partial F_i}{\partial x_k}(p + \sum_{j=1}^{k-1} v_j e_j + \theta e_k \right\| - \frac{\partial F_i}{\partial x_k}(p) \right\| \leqslant \frac{|v_k|\epsilon}{n}, \end{aligned}$$

and since $|v_j| \leqslant \|v\|$ for all j,

$$\frac{R_i(v)}{\|v\|} \leqslant \sum_{j=1}^{n} \frac{|v_j|\epsilon}{n\|v\|} \leqslant \sum_{j=1}^{n} \frac{\epsilon}{n} = \epsilon.$$

This proves the claim. □

Problem 5

Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$F(x, y, z) = (4x^2 + 4y^2 - 4z^2, x^2 + y^2 + z^2).$$

(a) Compute the Jacobian matrix of F.

(b) Consider the level set $F^{-1}(1, 1)$. Show that for all points (x, y, z) in this level set the Jacobian matrix F has maximal rank.

Solution.

(a) By definition,

$$(\mathcal{J}F)_{(x,y,z)} = \begin{bmatrix} 8x & 8y & -8z \\ 2x & 2y & 2z \end{bmatrix}.$$

(b) *Proof.* We prove by contradiction. Suppose $\mathcal{J}F$ at some $(x,y,z) \in F^{-1}(1,1)$ is not of maximal rank. Then it's of rank at most 1. In particular, the following two matrices would be singular:

$$\begin{bmatrix} 8x & 8y \\ 2x & 2y \end{bmatrix} \qquad \begin{bmatrix} 8y & -8z \\ 2y & 2z \end{bmatrix}.$$

The first one is, of course, always singular, but if the second one is singular then $z = 0$ (multiply second row by 4 and obtain $8z = -8z$). Then

$$4x^2 + 4y^2 = 1 \qquad x^2 + y^2 = 1,$$

clearly a contradiction. This finishes the proof. □

Problem 6

(1) Let $\beta := (x^2y + z)\mathrm{d}x + (xyz)\mathrm{d}y + (x + yz^2)\mathrm{d}z$, a 1-form on $\mathbb{R}^3$. Compute $\mathrm{d}\beta$.

(2) Let $P(x), Q(x)$ be smooth functions from $\mathbb{R}$ to $\mathbb{R}$. Consider the differential 1-form

$$\alpha := (P(x)y - Q(x))\mathrm{d}x + \mathrm{d}y$$

on $\mathbb{R}^2$. Let

$$\mu(x) := \exp \int_0^x P(t)\,\mathrm{d}t.$$

Prove that the 1-form

$$\mu\alpha = \mu(x)(P(x)y - Q(x))\mathrm{d}x + \mu(x)\mathrm{d}y$$

satisfies $\mathrm{d}(\mu\alpha) = 0$.

Solution.

(a)

$$\begin{aligned} \mathrm{d}\beta &= \left(2xy\mathrm{d}x + x^2\mathrm{d}y + \mathrm{d}z\right) \wedge \mathrm{d}x + (yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z) \wedge \mathrm{d}y + (\mathrm{d}x + z^2\mathrm{d}y + 2yz\mathrm{d}z) \wedge \mathrm{d}z \\ &= (yz - x^2)\mathrm{d}x \wedge \mathrm{d}y + (z^2 - xy)\mathrm{d}y \wedge \mathrm{d}z. \end{aligned}$$

(b) We will compute $\mathrm{d}(\mu\alpha)$ by brute force. Note that the x-partial of $\mu(x)(P(x)y - Q(x))$ does not matter because eventually it vanishes with $\mathrm{d}x \wedge \mathrm{d}x$. Meanwhile,

$$\frac{\partial}{\partial y}[\mu(x)(P(x)y - Q(x))] = \mu(x)P(x).$$

Therefore,

$$\begin{aligned} \mathrm{d}(\mu\alpha) &= \mu(x)P(x)\mathrm{d}y \wedge \mathrm{d}x + \mu'(x)\mathrm{d}x \wedge \mathrm{d}y \\ [\text{chain rule}] &= \mu(x)P(x)\mathrm{d}y \wedge \mathrm{d}x + \mu(x)P(x)\mathrm{d}x \wedge \mathrm{d}y = 0. \end{aligned}$$

□

Problem 7

Prove the Picard-Lindelöf Theorem: if $U \subset \mathbb{R}^m$ is open and $F : U \to \mathbb{R}^m$ is locally Lipschitz, then, given $p \in U$, there exists a locally unique solution to the IVP $\gamma'(t) = F(\gamma(t))$ defined on $(a, b) \in \mathbb{R}$ with $x(t_0) = y_0$.

Proof. For convenience let $t_0 = 0$ (the generic case can be obtained via the integral equation once the case $t_0 = 0$ is proven). WLOG assume F is Lipschitz with constant L on all of U. Pick $r > 0$ such that $N := \overline{B(y_0, r)} \subset U$. Since N is closed and bounded in $\mathbb{R}^m$, it is compact, on which the continuous image $F(N)$ is also compact. Hence there exists $M \in \mathbb{R}$ such that $\|F(x)\| \leqslant M$ for all $x \in N$.
Now pick $\tau > 0$ sufficiently small such that $\tau < \min(r/M, 1/L)$. We claim that

(1) there exists $\gamma : (-\tau, \tau) \to N$ differentiable with $\gamma'(t) = F(\gamma(t))$, and

(2) such γ is unique.

Notice that solving the IVP $\gamma'(t) = F(\gamma(t)), \gamma(0) = y_0$ is equivalent to solving

$$\gamma(t) = \gamma(0 + \int_0^t F(\gamma(s))\, \mathrm{d}s.$$

Since the space $(C^0([-\tau, \tau], N), \mathrm{d}_{\sup})$ is Banach, if we can show that

$$(\Phi(\gamma))(t) := y_0 + \int_0^t F(\gamma(s))\, \mathrm{d}s$$

is a contraction, then by the Banach contraction mapping theorem, there exists a fixed point which would solve our IVP. Clearly $\Phi(\gamma)$ is continuous, and for $t \in [-\tau, \tau]$,

$$\|\Phi(\gamma)(t) - y_0\| = \left\| \int_0^t F(\gamma(s))\, \mathrm{d}s \right\| \leqslant M|t - 0| \leqslant M\tau = M \cdot min(r/M, 1/L) \leqslant r,$$

so indeed $\Phi(\gamma)(t)$ is always an element of $C^0([-\tau, \tau], N)$. Now we show that Φ is actually a contraction with constant $\tau L < 1$. If $\gamma, \sigma \in C^0([-\tau, \tau], N)$, then

$$\begin{aligned} d(\Phi(\gamma), \Phi(\sigma)) &= \sup_{t \in [-\tau, \tau]} \left\| y_0 + \int_0^t F(\gamma(s))\, \mathrm{d}s - y_0 - \int_0^t F(\sigma(s))\, \mathrm{d}s \right\| \\ &= \sup_{t \in [-\tau, \tau]} \left\| \int_0^t F(\gamma(s)) - F(\sigma(s))\, \mathrm{d}s \right\| \\ &\leqslant \sup_{t \in [-\tau, \tau]} |\, t\, | \cdot \sup_{s \in [\tau, \tau]} \|F(\gamma(s)) - F(\sigma(s))\| \\ &\leqslant \tau \cdot L \sup_{s \in [-\tau, \tau]} \|\gamma(s) - \sigma(s)\| = \tau L \cdot \mathrm{d}_{\sup}(\gamma, \sigma). \end{aligned}$$

Now we can invoke the Banach contraction mapping theorem and conclude that there exists a solution to the IVP. Uniqueness follows from one of our HWs, in which we've shown that if $\gamma : (a, b) \to U$ and $\sigma : (a', b') \to U$ are two solutions to the IVP then γ and σ must agree on $(a, b) \cap (a', b')$. Local uniqueness still holds. □

Alternate Problem 7

Problem: 7(a)

Let $c > 0$. For any complex number s, show that the improper Riemann integral

$$\int_1^\infty t^{s-1}e^{-ct}\,\mathrm{d}t$$

converges. If the real part of s is greater than 0, prove the improper Riemann integral

$$\int_0^\infty t^{s-1}e^{-ct}\,\mathrm{d}t$$

converges. *When $c = 1$ this integral defines the Γ function.*

Proof. For the first part, it suffices to show that the integrals $\int_1^b t^{s-1}e^{-ct}\,\mathrm{d}t$ is Cauchy; equivalently, it suffices to show that when b_1, b_2 are sufficiently large, $\int_{b_1}^{b_2} t^{s-1}e^{-ct}\,\mathrm{d}t$ can be made arbitrarily small. Indeed, since $\lim_{t\to\infty}|t^s|/e^{ct/2} = \lim_{t\to\infty} t^{\mathfrak{Re}(s)}/e^{ct/2} = 0$ (as given by the hint), we are able to find b_1, b_2 large enough such that $|t^{s-1}| \leqslant e^{ct/2}$. Then we can bound the integral by

$$\int_{b_1}^{b_2} t^{s-1}e^{-ct}\,\mathrm{d}t \leqslant \int_{b_1}^{b_2} |t^{s-1}e^{-ct}|\,\mathrm{d}t \leqslant \int_{b_1}^{b_2} e^{-ct/2}\,\mathrm{d}t.$$

The last term is much nicer, as

$$\int_1^\infty e^{-ct/2}\,\mathrm{d}t = -\frac{2}{c}e^{-ct/2}\bigg|_1^\infty = \frac{2e^{-c/2}}{c} < \infty,$$

which implies $\int_{b_1}^{b_2} e^{-ct/2}\,\mathrm{d}t \to 0$ as $\min(b_1, b_2) \to \infty$. This proves the first claim.

For the second claim, we only need to verify that $\int_0^1 t^{s-1}e^{-ct}\,\mathrm{d}x$ is finite. Since $c > 0$ and $t > 0$, $e^{-ct} < 1$, so

$$\int_0^1 t^{s-1}e^{-ct}\,\mathrm{d}t < \int_0^1 t^{s-1}\,\mathrm{d}t \leqslant \int_0^1 |t^{s-1}|\,\mathrm{d}t = \int_0^1 t^{\mathfrak{Re}(s-1)}\,\mathrm{d}t.$$

By assumption $\mathfrak{Re}(s) > 0$ so $\mathfrak{Re}(s-1) > -1$. Even Calc I students would realize that this integral is finite. □

Problem: 7(b)

For any complex number s with real part > 1, we want to define (a variant of) the Riemann ζ function using the improper Riemann integral formula

$$\xi(s) := \int_0^\infty t^{s/2-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right)\mathrm{d}t.$$

Show that this improper Riemann integral exists and is given by

$$\xi(s) = 2\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

where

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t \qquad \zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s}.$$

Proposition

For this problem, you may assume the following statement on nets. Let $(A, \leq)$ be a directed set and consider a sequence $f_n(a)$ of a real-valued nets $f_n : A \to \mathbb{R}$. Assume that

(1) for each n, the net $f_n(a)$ converges (in a) to a limit $L_n \in \mathbb{R}$ and

(2) the nets f_n converge uniformly (as a sequence of functions $A \to \mathbb{R}$) to some net $f : A \to R$.

Then L_n converge to some $L \in \mathbb{R}$ and $\lim_a f(a)$ exists and equals L. In other words,

$$\lim_a \lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \lim_a f_n(a).$$

Proof. Following the hint, we will first consider

$$\int_a^b t^{s/2-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right)\mathrm{d}t$$

and then let $a \to 0$ and $b \to \infty$. By the first problem, $\sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$ is the limit of a uniformly convergent series of continuous functions and is therefore continuous. Thus the entire integrand is continuous and the integral from a to b is well-defined.

In addition, since $\int$ and $\sum$ for a uniformly convergent series are interchangeable,

$$\int_a^b t^{s/2-1}\left(\sum_{n=-1}^{-\infty} e^{-\pi t n^2} + \sum_{n=1}^{\infty} e^{-\pi t n^2}\right)\mathrm{d}t = 2\sum_{n=1}^{\infty}\int_a^b t^{s/2-1}e^{-\pi t n^2}\,\mathrm{d}t.$$

Now define A to be the set of pairs of real numbers (a, b) with $0 < a < b$ where $\leq$ is defined as $(a, b) \leq (a', b')$ if $a' < a < b < b'$. Define

$$f_k(a, b) := 2\sum_{n=1}^{k}\int_a^b t^{s/2-1}e^{-\pi t k^2}\,\mathrm{d}t.$$

By part (a), since $\mathfrak{Re}(s/2) > 0$, $\lim_{(a,b)\to(0,\infty)} f_k(a, b)$ exists for any k. The net condition (1) is satisfied.

To show our construction of $f_k(a, b)$ also satisfy condition (2), we need to show $\{f_k\}$ is uniformly Cauchy. To do this, we need a u-substitution $u = \pi n^2 t$. For $M > N$ and arbitrary (a, b), this gives

$$\begin{aligned} f_M(a,b) - f_N(a,b) &= 2\sum_{n=N}^{M}\int_a^b t^{s/2-1}e^{-\pi t n^2}\,\mathrm{d}t \\ &= 2\sum_{n=N}^{M}\int_{\pi n^2 a}^{\pi n^2 b}\left(\frac{u}{\pi n^2}\right)^{s/2-1} e^{-u}\frac{\mathrm{d}u}{\pi n^2} \\ &= 2\pi^{-s/2}\cdot\int_{\pi n^2 a}^{\pi n^2 b} u^{s/2-1}e^{-u}\,\mathrm{d}u \cdot \sum_{n=N}^{M}\frac{1}{n^s} \\ &< 2\pi^{-s/2}\Gamma(s/2)\sum_{n=N}^{M}\frac{1}{n^s}. \end{aligned}$$

We know that if $\mathfrak{Re}(s) > 1$ then $\sum_{n-1}^{\infty} \frac{1}{n^s}$ converges, so indeed the last term can be made arbitrarily small. This proves that the f_k's are uniformly Cauchy. By the proposition,

$$\lim_{(a,b)\to(0,\infty)} \lim_{k\to\infty} f_k(a,b) = \lim_{k\to\infty} \lim_{(a,b)\to(0,\infty)} f_k(a,b).$$

Evaluating the RHS is highly analogous to the computation above, except this time we actually have $\zeta(s)$ for the last term. Therefore,

$$\xi(s) = 2\sum_{n=1}^{\infty} \int_0^{\infty} t^{s/2-1} e^{-\pi t n^2}\, \mathrm{d}t = 2\pi^{-s/2}\Gamma(s/2)\zeta(s). \qquad \square$$

Problem: 7(c)

Show that the improper Riemann integral

$$\int_1^{\infty} t^{-1}(t^{s/2} + t^{(1-s)/2}) \left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1 \right) \mathrm{d}t$$

makes sense for all complex s.

Proof. We will again adopt the notion of convergent nets here. Just like before, we can rewrite the doubly-infinite series minus 1 as 2 times a singly-infinite series and we can also interchange the summation and integral. Define A to be the set of real numbers > 1 and $b \le b'$ if $b < b'$, and define

$$f_k(b) := 2\int_1^b t^{-1}(t^{s/2} + t^{(1-s)/2}) \left(\sum_{n=1}^{k} e^{-\pi t n^2} \right) \mathrm{d}t.$$

Again, by part (a), for any k, $\lim_{b\to\infty} f_k(b)$ exists so the net condition (1) is satisfied.

Now we verify that $f_k(b)$ form a uniformly Cauchy sequence as $k \to \infty$. For $M > N$,

$$\begin{aligned} f_M(b) - f_N(b) &= 2\sum_{n=N}^{M} \int_1^b (t^{s/2-1} + t^{(1-s)/2}) e^{-\pi t n^2}\, \mathrm{d}t \\ &= 2\sum_{n=N}^{M} e^{-\pi t n^2/2} \int_1^b (t^{s/2-1} + t^{(1-s)/2}) e^{-\pi t n^2/2}\, \mathrm{d}t \\ &< 2\sum_{n=N}^{M} e^{-\pi n^2/2} \int_1^b A t^K e^{-\pi t n^2/2}\, \mathrm{d}t \\ &\leqslant 2\sum_{n=N}^{M} e^{-\pi n^2/2} \underbrace{\int_1^b A t^{(K+1)-1} e^{t\cdot(-\pi n^2/2)}\, \mathrm{d}t}_{\text{finite for each } n} \end{aligned}$$

where A and K are some constants, as one between $t^{s/2-1}$ and $t^{(1-s)/2}$ will surely decay and the total sum is clearly possible to be bounded by At^K. Since we are looking at a finite sum, we can just leave the integral like that, as long as each one is finite by part (a). It remains to show that $\sum_{n=N}^{M} e^{-\pi m^2/2}$ is finite, which we've shown all the way back in problem 1(b). Therefore the net condition (2), and the proposition gives

$$\int_1^{\infty} t^{-1}(t^{s/2} + t^{(1-s)/2}) \left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1 \right) \mathrm{d}t = 2\sum_{n=1}^{\infty} \int_1^{\infty} t^{-1}(t^{s/2} + t^{(1-s)/2}) e^{-\pi t n^2}\, \mathrm{d}t. \qquad \square$$

Problem: 7(d)

Show that for complex s with $\mathfrak{Re}(s) > 1$, we have

$$\xi(s) = -\frac{2}{s(1-s)} + \int_1^\infty t^{-1}(t^{s/2} + t^{(1-s)/2})\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt.$$

Proof. We first split $\int_0^\infty$ into $\int_0^1 + \int_1^\infty$:

$$\xi(s) = \int_0^1 t^{s/2-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt + \int_1^\infty t^{s/2-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt. \tag{♏}$$

Using u-substitution $u := 1/t$ with $du = -t^{-2}dt$ and the result from problem 1(c), we have

$$\begin{aligned}
\int_0^1 t^{s/2-1}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - 1\right) dt &= \int_\infty^1 u^{1-s/2}\left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2/u} - 1\right)(-t^2)\, du \\
&= \int_1^\infty u^{1-s/2}\left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2/u} - 1\right) u^{-2}\, du \\
&= \int_1^\infty u^{-1-s/2}\left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2/u} - 1\right) du \\
[\text{prob } 1(\text{c})] &= \int_1^\infty u^{-1-s/2}\left(\sum_{n=-\infty}^{\infty} \sqrt{u}e^{-\pi n^2 u} - 1\right) du \\
&= \int_1^\infty u^{(-1-s)/2}\left(\sum_{n=-\infty}^{\infty} e^{-\pi u n^2} - \frac{1}{\sqrt{u}}\right) du \\
&= \int_1^\infty t^{(-1-s)/2}\left(\sum_{n=-\infty}^{\infty} e^{-\pi t n^2} - \frac{1}{\sqrt{t}}\right) dt.
\end{aligned} \tag{♈}$$

Substituting (♈) into (♏), we see that the difference between their sum and the (big) integral given in the problem is

$$\begin{aligned}
\int_1^\infty t^{(-s-1)/2}\left(1 - \frac{1}{\sqrt{t}}\right) dt &= \int_1^\infty t^{(-s-1)/2}\, dt - \int_1^\infty t^{-s/2-1}\, dt \\
&= \frac{2}{s-1} - \frac{2}{s} = -\frac{2}{s(1-s)}.
\end{aligned}$$

This proves the claim. □

Problem: 7(e)

Show that the zeta function satisfies the function equation

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

You may assume that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \qquad \text{for } 0 < \mathfrak{Re}(s) < 1$$

and that

$$\Gamma(s/2)\Gamma((s+1)/2) = \frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s).$$

Proof. Notice from part (d) that $\xi(s) = \xi(1-s)$. Thus,

$$\zeta(s) = \frac{\xi(s)\pi^{s/2}}{2\Gamma(s/2)} \text{ and } \zeta(1-s) = \frac{\xi(1-s)\pi^{(1-s)/2}}{2\Gamma((1-s)/2)} = \frac{\xi(s)\pi^{(1-s)/2}}{2\Gamma((1-s)/2)}.$$

Therefore we can relate $\zeta(s)$ with $\zeta(1-s)$ by

$$\zeta(s) = \frac{\xi(s)\pi^{s/2}}{2\Gamma(s/2)} = \frac{\xi(s)\pi^{(1-s)/2}}{2\Gamma((1-s)/2)} \cdot \frac{\pi^{(2s-1)/2}\Gamma((1-s)/2)}{\Gamma(s/2)}.$$

Since $\sin(\pi s/2) = \dfrac{\pi}{\Gamma(s/2)\Gamma(1-s/2)}$, the cyan term becomes

$$\frac{\pi^{(2s-1)/2}\Gamma((1-s)/2)}{\Gamma(s/2)} = \sin(\pi s/2)\pi^{(2s-3)/2}\Gamma(1-s/2)\Gamma((1-s)/2).$$

Using the second identity given, $\Gamma(1-s/2)\Gamma((1-s)/2) = \dfrac{\sqrt{\pi}}{2^{-s}}\Gamma(1-s)$. Therefore,

$$\begin{aligned}\zeta(s) &= \zeta(1-s)\cdot\sin(\pi s/2)\cdot\pi^{(2s-3)/2}\cdot\frac{\sqrt{\pi}}{2^{-s}}\cdot\Gamma(1-s)\\ &= 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).\end{aligned}$$

□

Problem: 7(f)

Given the above extension of $\zeta(s)$ to the complex plane, show that $\zeta(s) \neq 0$ unless $s = -2k$ or $\mathfrak{Re}(s) = 1/2$.

Proof. I have discovered a truly remarkable proof which this margin is too small to contain. Alas, how could I possibly cram it into the remaining 1/3 page below? □