MATH 425b Midterm 1

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Problem 1

(a) Let V be a vector space. A norm is a map $\|\cdot\| : V \to [0,\infty)$ satisfying the following:

(1) non-degeneracy: $\|v\| \geqslant 0$ and $\|v\| = 0$ if and only if $v = 0$,

(2) absolute homogeneity: $\|\lambda x\| = |\lambda|\|x\|$, and

(3) subadditivity: $\|u+v\| \leqslant \|u\| + \|v\|$,

where $\lambda \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and u, v are arbitrary elements of V. A vector space equipped with such a norm is called a **normed vector space**, written $(V, \|\cdot\|)$. A normed space is **Banach** if it is complete with respect to that norm, i.e., every Cauchy sequence converges.

(b) Since $\sum\limits_{k=1}^{\infty} \|v_k\|$ converges, in particular we have that it is also Cauchy. Let $\epsilon > 0$ be given. There exists some $N \in \mathbb{N}$ such that if $m > n \geqslant N$ then $\sum\limits_{k=m}^{n} \|v_k\| < \epsilon$. Then, for the same N, if $m > n \geqslant N$, we have

$$\left\| \sum_{k=1}^{n} v_k - \sum_{k=1}^{m} v_k \right\| = \left\| \sum_{k=m}^{n} v_k \right\| \leqslant \sum_{k=m}^{n} \|v_k\| < \epsilon,$$

where the $\leqslant$ is given by triangle inequality. Therefore the series of v_k forms a Cauchy sequence in a Banach space, and thus it converges. □

Problem 2

(a) Let $\epsilon > 0$ be given. By the uniform convergence of $\{f_n\}$, there exists $N \in \mathbb{N}$ such that $\|f_n - f\|_{\sup} < \epsilon/(b-a)$ whenever $n \geqslant N$. Therefore,

$$\begin{aligned}
|F_n(x) - F(x)| &= \left| \int_a^x f_n(\tilde{t}) \,\mathrm{d}\tilde{t} - \int_a^x f(\tilde{t}) \,\mathrm{d}\tilde{t} \right| \\
&= \left| \int_a^x f_n(\tilde{t}) - f(\tilde{t}) \,\mathrm{d}\tilde{t} \right| \\
&\leqslant \int_a^x \left| f_n(\tilde{t}) - f(\tilde{t}) \right| \mathrm{d}\tilde{t} \\
&< (x-a)\frac{\epsilon}{b-a} \leqslant \epsilon,
\end{aligned}$$

and the uniform convergence of $\{F_n\}$ to F follows.

(b) Recall the example given on one of the homeworks: let $f_n(x) := \sqrt{x^2 + 1/n}$ and $f(x) := |x|$ on $[1, 1]$. It is clear that f_n's are differentiable. For $n \geqslant 0$,

$$\sqrt{x^2 + 1/n} - |x| = \sqrt{x^2 + 1/n} - x \leqslant (x + 1/n^2) - x = 1/n^2$$

regardless of the value of f, and when $n < 0$ the argument can be treated similarly. Thus $f_n \to f$ uniformly but clearly f is not differentiable.

Problem 3

(a) The **exponential growth rate**, e.g.r., is defined to be $\alpha := \limsup_{k\to\infty} \sqrt[k]{|a_k|}$.

(b) If $\alpha < 1$, there exists $\beta \in (\alpha, 1)$. By the definition of lim sup there exists a sufficiently large N such that if $n \geqslant N$ then $\sqrt[n]{|a_n|}$ is close enough to α and thus $< \beta$. Now construct a sequence $(\beta^n, \beta^{n+1}, \dots)$. Since, for $n \geqslant N$, $|a_n| < \beta^n$ and $\sum_{k=n}^{\infty} \beta^n$ converges, so does the series of a_k. □

(c) If $\alpha > 1$, there exists $\beta \in (1, \alpha)$. By definition,

$$\lim_{n\to\infty} \sup_{\tilde{k}\geqslant n} \sqrt[\tilde{k}]{|a_{\tilde{k}}|} = \alpha > \beta,$$

so $\sup_{\tilde{k}\geqslant n} \sqrt[\tilde{k}]{|a_{\tilde{k}}|}$ is always greater than β, regardless of n. This means that there exists infinitely many $\tilde{k}$ such that $\sqrt[\tilde{k}]{|a_{\tilde{k}}|} > \beta$, i.e., $|a_{\tilde{k}}| > \beta^{\tilde{k}}$. Therefore $\lim_{k\to\infty} a_k \neq 0$ and the series does not converge. □

Problem 4

(a) A net f is said to be **Cauchy** if, for any $\epsilon > 0$, there exists $a_0 \in A$ such that $d(f(a_1), f(a_2)) < \epsilon$ whenever $a_0 \preceq a_1$ and $a_0 \preceq a_2$.

(b) Let $f : A \to X$ be Cauchy. By Cauchy-ness, there exists $a_1 \in A$ such that if $a_0 \preceq a'$ and $a_0 \preceq a''$ then $d(f(a'), f(a'')) < \epsilon = 1$. Now pick $a_2 \in A$ such that $a_1 \preceq a_2$ and $d(f(a'), f(a'')) < 1/2$ whenever $a_2 \preceq a'$ and $a_2 \preceq a''$. (This $(a_1 \preceq a_2)$ can always be made possible by the upper bound property of a directed set.) Keep decreasing ϵ to $1/3, 1/4 \dots$ and we obtain a sequence $a_1 \preceq a_2 \preceq a_3 \dots$ that satisfies the corresponding Cauchy requirement. Define a sequence $\{x_n\} \subset X$ by $x_n := f(a_n)$. Immediately we see that $\{x_n\}$ is Cauchy in X: for $\epsilon > 0$, picking $N > 1/\epsilon$ gives $1/N < \epsilon$ and so for all $m, n \geqslant N$, $a_N \preceq a_m$ and $a_N \preceq a_m$, which implies $d(f(a_m), f(a_n)) = d(x_m, x_n) < 1/N < \epsilon$.

By the completeness of X, $\{x_n\} \to L$ for some $L \in X$. Given $\epsilon > 0$, there exists N_1 such that $d(x_n, L) < \epsilon/2$ whenever $n \geqslant N_1$. There also exists N_2 such that $(a_{N_2} \preceq a', a'' \in A \implies d(f(a'), f(a'')) < \epsilon/2)$. Define $N := \max\{N_1, N_2\}$. Then, for all $a \in A$ with $a_N \preceq a$, we have

$$d(f(a) - L) \leqslant d(f(a) - f(a_N)) + d(f(a_N) - L) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which completes the proof that f is a convergent net. □

Problem 5

Since Y is complete, for each x, the sequence $\{f_n(x)\}_{n\geqslant 1}$ is not only Cauchy but also convergent. Therefore we can define f to be the function to which $\{f_n\}$ converges pointwise.

Now let $\epsilon > 0$ be given. By the uniform Cauchy-ness, there exists $N \in \mathbb{N}$ such that $\|f_m - f_n\|_{\sup} < \epsilon/2$ whenever $m, n \geqslant N$, i.e., $|f_m(x) - f_n(x)| < \epsilon/2$ for all x in the domain. Let $n > N$. Then, for every $m > N$ we have

$$|f_n(x) - f(x)| \leqslant |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + |f_m(x) - f(x)|$$

where the second term can be made arbitrarily small, in particular $< \epsilon/2$ by setting m large, but we don't care about m eventually since it does not appear on the LHS; its mere existence is to show that $|f_n(x) - f(x)|$ can indeed be bounded by ϵ. Hence $f_n \to f$ uniformly. $\square$