# MATH 425b Midterm II

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**Problem 1**

Prove the *Weierstraß Approximation Theorem*. In particular, prove that any $f \in C^0([0,1],\mathbb{R})$ is the uniform limit of a sequence of Bernstein polynomials.

*Proof.* Let $f \in C^0([0,1],\mathbb{R})$ be given. Define

$$p_n(x) := \sum_{k=0}^{n} f(k/n) r_k(x) \text{ where } r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

(the Bernstein polynomials). We first derive some important identities that will become useful.

(1) Recall from binomial expansion,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{1}$$

Differentiating both sides with respect to $x$ and then multiplying by $x$ give

$$nx(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^k y^{n-k}. \tag{2}$$

Differentiating (1) twice and multiplying both sides by $x^2$ give

$$n(n-1)x^2(x+y)^{n-2} = \sum_{k=0}^{n} \binom{n}{k} k(k-1) x^k y^{n-k}. \tag{3}$$

Setting $y = 1-x$, (1), (2), and (3) give

$$\sum_{k=0}^{n} r_k(x) = 1, \sum_{k=0}^{n} k r_k(x) = nx, \text{ and } \sum_{k=0}^{n} k(k-1) r_k(x) = n(n-1)^2.$$

(2) *The variance of binomial distribution is* $nx(1-x)$:

$$\begin{aligned}\sum_{k=0}^{n} (k-nx)^2 r_k(x) &= \sum_{k=0}^{n} (k^2 - 2nxk + n^2x^2) r_k(x) \\ &= \sum_{k=0}^{n} k^2 r_k(x) - 2nx \sum_{k=0}^{n} k r_k(x) + n^2x^2 \sum_{k=0}^{n} r_k(x) \\ &= [n(n-1)^2 + nx] - 2n^2x^2 + n^2x^2 \\ &= nx(1-x).\end{aligned}$$

**Back to the main proof.** Let $\epsilon > 0$ be given. Since $[0,1]$ is compact and $f$ continuous, it is uniformly continuous on $[0,1]$. Therefore, there exists $\delta > 0$ such that

$$|x-y| < \delta \implies f(x) - f(y) < \frac{\epsilon}{2}.$$

In addition, $f([0,1])$ is compact (thus bounded), so there exists $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [0,1]$. Let $N \geqslant M/(\epsilon\delta^2)$ be a sufficiently large integer. Since $\sum_{k=0}^{n} r_k(x) = 1$, $f(x)$ is the same as $\sum_{k=0}^{n} f(x) r_k(x)$. Then,

$$\begin{aligned}
|p_n(x) - f(x)| &= \left| \sum_{k=0}^{n} f(k/n) r_k(x) - \sum_{k=0}^{n} f(x) r_k(x) \right| \\
&= \left| \sum_{k=0}^{n} [f(k/n) - f(x)] r_k(x) \right| \\
&\leqslant \sum_{\substack{k=0 \\ |x-\frac{k}{n}|<\delta}}^{n} |[f(k/n) - f(x)] r_k(x)| + \sum_{\substack{k=0 \\ |x-\frac{k}{n}|\geqslant\delta}}^{n} |[f(k/n) - f(x)] r_k(x)| && \text{denote by } \sum_1 \,\&\, \sum_2 \\
&< \sum_1 (\epsilon/2) r_k(x) + \sum_2 2M r_k(x) \cdot 1 && \epsilon/2 \text{ by unif. cont; } 2M \text{ by boundedness} \\
&\leqslant \frac{\epsilon}{2} + \sum_2 2M r_k(x) \cdot \frac{|k-nx|^2}{(n\delta)^2} && \text{since } 1 \leqslant \left(\frac{|k/n_0 x|}{\delta}\right)^2 = \frac{|k-nx|^2}{(n\delta)^2} \\
&\leqslant \frac{\epsilon}{2} + \frac{2M}{n^2\delta^2} \sum_{k=1}^{n} |k-nx|^2 r_k(x) \\
&= \frac{\epsilon}{2} + \frac{2M}{n^2\delta^2} nx(1-x) = \frac{\epsilon}{2} + \frac{2Mx(1-x)}{n\delta^2} \\
&\leqslant \frac{\epsilon}{2} + \frac{M}{2n\delta^2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } n \geqslant N. && \text{simple calculus:} x(1-x) \leqslant \frac{1}{4}
\end{aligned}$$

...and we are done with the proof. □

**Problem 2**

(a) Let $(X,d)$ be a metric space. Define what it means for $f : X \to X$ to be a contraction.

(b) State and prove the *Banach contraction mapping theorem.*

**Definition: 2(a)**

$f$ is a **contraction** if there exists $k < 1$ such that $d(f(x), f(y)) \leqslant kd(x,y)$ for all $x, y \in X$. If such $k < 1$ does not exist but still $d(f(x), f(y)) < d(x,y)$, we call $f$ a **weak contraction**.

**Theorem: Banach contraction mapping theorem, 2(b)**

Let $(X,d)$ be complete and let $f : X \to X$ be a contraction. Then $f$ has a unique fixed point.

*Proof.* Pick any $x_0 \in X$. Define iteratively $x_n = f(x_{n-1})$ (so $x_1 = f(x_0)$ and so on). Let $\ell < 1$ be the "contraction constant" for $f$, i.e., $d(f(x), f(y)) \leqslant kd(x,y)$ for all $x, y \in X$. It follows that

$$d(x_n, x_{n+1}) \leqslant \ell d(x_{n-1}, x_n) \leqslant \cdots \leqslant \ell^n d(x_0, x_1).$$

We now show that the sequence $\{x_n\}$ is Cauchy. Let $\epsilon > 0$ be given. For any $m, n$, we have (assuming WLOG $m < n$)

$$\begin{aligned} d(x_m, x_n) &\leqslant \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leqslant \sum_{k=m}^{\infty} d(x_k, x_{k+1}) \\ &\leqslant \sum_{k=0}^{\infty} d(x_m, x_{m+1}) \ell^k \\ &\leqslant \frac{\ell^m d(x_0, x_1)}{1 - \ell}. \end{aligned}$$

To bound this by $\epsilon$, we simply need to choose $N$ large enough such that $\ell^N d(x_0, x_1)/(1-\ell) < \epsilon$. Then for any $m, n \geqslant N$ we obtain the desired inequality. Hence $\{x_n\}$ is Cauchy and, since $(X, d)$ is complete, convergent to $x \in X$, say. Then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$$

where the second last = is by continuity of $f$. Hence $x$ is a fixed point. Uniqueness is immediate as if $x, y$ are both fixed points then

$$x = f(x), y = f(y) \implies d(x,y) = d(f(x), f(y)) > kd(x,y)$$

unless $d(x,y) = 0$, i.e., fixed point is unique. ☐

**Problem 3**

Define what it means for a subset $A$ of a metric space $(X, d)$ to be dense. Prove the $\delta$-*density lemma*, i.e., if $X$ is compact and $A = \{a_1, a_2, \dots\}$ is countably dense then, for $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in X$, for some $1 \leqslant i \leqslant N$ we have $d(x, a_i) < \delta$.

**Definition: Dense subsets**

Let $(X, d)$ be a metric space. We say $A \subset X$ is **dense** in $X$ if $\overline{A} = X$. Equivalently, $A$ is dense in $X$ if, for any $\epsilon > 0$ and for all $x \in X$, there exists $a \in A$ such that $d(x, a) < \epsilon$.

*Proof of the $\delta$-density lemma.* (Without Dini's theorem unfortunately; I wanted to secure the points...) Consider the open covering $\bigcup_{i\geqslant 1} D_i$ where $D_i := B(a_i, \delta)$, i.e., the $\delta$-ball around $a_i$. This is an open covering of $X$ because $\{a_i\}$ is dense in $X$ (and so for all $x \in X$, there exists $a_i \in \{a_i\}$ with $d(x, a_i) < \delta$, i.e., $x \in B(a_i, \delta)$). By the compactness of $X$, this open covering can be reduced to a finite subcovering, say $\bigcup_{i=1}^{N} D_i$. Then any $x \in X$ is included in a $\delta$-ball around some $a_i \in \{a_1, \dots, a_N\}$, and this proves the claim. ☐

**Problem 4**

Let $f : \mathbb{R} \to \mathbb{C}$ be a *suitably nice* function. Write down the integral defining the Fourier transform $\hat{f}(\xi)$ for $\xi \in \mathbb{R}$. Using Fourier transforms, find one solution $f(t)$ to the differential equation (and check explicitly)

$$f'(t) + 2f(t) = \cos(2\pi t).$$

**Solution**

Firstly, for $f \in \mathbb{R} \to \mathbb{C}$, we have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, \mathrm{d}x.$$

Since $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$2\cos\theta = \cos\theta + \cos(-\theta) + i\sin(\theta) + i\sin(-\theta) = e^{i\theta} + e^{-i\theta}.$$

Therefore, $\cos(2\pi t) = (e^{2\pi i t} + e^{-2\pi i t})/2$. Define this to be $g(t)$. It follows that

$$\begin{aligned}
\hat{g}(\xi) &= \int_{-\infty}^{\infty} g(t) e^{-2\pi i \xi t} \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{2\pi i t} e^{-2\pi i \xi t}/2 \, \mathrm{d}t + \int_{-\infty}^{\infty} e^{-2\pi i t} e^{-2\pi i \xi t}/2 \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{2\pi i t(1-\xi)}/2 \, \mathrm{d}t + \int_{-\infty}^{\infty} e^{2\pi i t(-1-\xi)}/2 \, \mathrm{d}x \\
&= \frac{\delta(1-\xi) + \delta(-1-\xi)}{2}.
\end{aligned}$$

Recall from HW10 that if $f'(t) + 2f(t) = g(t)$ then

$$[2\pi i \xi + 2] \hat{f}(\xi) = \hat{g}(\xi) \implies \hat{f}(\xi) = \frac{\delta(1-\xi) + \delta(-1-\xi)}{4\pi i \xi + 4}.$$

Using the *Fourier inversion formula* we have

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} \, \mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} \delta(1-\xi) \frac{e^{2\pi i \xi x}}{4\pi i \xi + 4} \, \mathrm{d}\xi + \int_{-\infty}^{\infty} \delta(-1-\xi) \frac{e^{2\pi i \xi x}}{4\pi i \xi + 4} \, \mathrm{d}\xi \\
&= \left.\frac{e^{2\pi i \xi x}}{4\pi i \xi + 4}\right|_{\xi=1} + \left.\frac{e^{2\pi i \xi x}}{4\pi i \xi + 4}\right|_{\xi=-1} \\
&= \frac{e^{2\pi i x}}{4 + 4\pi i} + \frac{e^{-2\pi i x}}{4 - 4\pi i},
\end{aligned}$$

which is is the solution to the inhomogeneous system. Verification:

$$\begin{aligned}
f'(x) + 2f(x) &= \frac{2\pi i e^{2\pi i x}}{4 + 4\pi i} - \frac{2\pi i e^{-2\pi i x}}{4 - 4\pi i} + \frac{2e^{2\pi i x}}{4 + 4\pi i} + \frac{2e^{-2\pi i x}}{4 - 4\pi i} \\
&= \frac{(2 + 2\pi i) e^{2\pi i x}}{4 + 4\pi i} + \frac{(2 - 2\pi i) e^{-2\pi i x}}{4 - 4\pi i} \\
&= \frac{e^{2\pi i x} + e^{-2\pi i x}}{2} = \cos(2\pi x).
\end{aligned}$$

**Problem 5**

Define $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ by setting $f(x) = (1/2 - x)^2$ for $x \in [0,1]$ and extending $f$ periodically to all of $\mathbb{R}$.

(a) Compute the Fourier coefficients $\hat{f}(n)$ for all $n \in \mathbb{Z}$ and write down the corresponding Fourier series as a doubly infinite sum of exponentials.

(b) Prove that the Fourier series converge uniformly to $f$.

(c) Prove $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

**Solution: 5(a)**

For $n = 0$ it is clear enough that

$$\hat{f}(0) = \int_0^1 e^0 (1/2 - \theta)^2 \, \mathrm{d}\theta = -\frac{1}{3}(1/2 - \theta)^3 \Big|_{\theta=0}^1 = \frac{1}{12}.$$

For $n \neq 0$, we first compute $\hat{f}(n)$ for $n \in [0,1]$. First notice that

$$\hat{f}(n) = \int_0^1 f(\theta) e^{-2\pi i n \theta} \, \mathrm{d}\theta = \int_0^1 e^{-2\pi i n\theta}/4 \, \mathrm{d}\theta - \int_0^1 e^{-2\pi i n \theta}\theta \, \mathrm{d}\theta + \int_0^1 e^{-2\pi i n\theta}\theta^2 \, \mathrm{d}\theta.$$

With Euler's identity, the first integral becomes

$$\begin{aligned}
\frac{1}{4}\int_0^1 \cos(2\pi n\theta) - i\sin(2\pi n\theta) \, \mathrm{d}\theta &= \frac{1}{4}\left[\frac{\sin(2\pi n\theta)}{2\pi n} + \frac{i\cos(2\pi n\theta)}{2\pi n}\right]_{\theta=0}^1 \\
&= \frac{1}{8\pi n}(0 + i - 0 - i) = 0.
\end{aligned}$$

The second one, using integration by parts ($u := \theta$ and $\mathrm{d}u := e^{-2\pi i n\theta}\mathrm{d}\theta$) and Euler's identity, becomes

$$\int_0^1 e^{-2\pi i n\theta}\theta \, \mathrm{d}\theta = -\frac{\theta e^{-2\pi i n\theta}}{2\pi i n}\Big|_{\theta=0}^1 + \int_0^1 \frac{e^{-2\pi i n\theta}}{2\pi i n} \, \mathrm{d}\theta = -\frac{1}{2\pi i n} + 0 = -\frac{1}{2\pi i n}.$$

Likewise, the third term becomes

$$\begin{aligned}
\int_0^1 e^{-2\pi i n\theta}\theta^2 \, \mathrm{d}\theta &= \frac{\theta^2 e^{-2\pi i n\theta}}{-2\pi i n}\Big|_{\theta=0}^1 + \int_0^1 \frac{2\theta e^{-2\pi i n\theta}}{2\pi i n} \, \mathrm{d}\theta \\
&= -\frac{1}{2\pi i n} + \frac{1}{\pi i n}\int_0^1 \theta e^{-2\pi i n\theta} \, \mathrm{d}\theta = -\frac{1}{2\pi i n} + \frac{1}{2\pi^2 n^2}.
\end{aligned}$$

Therefore,

$$\hat{f}(n) = 0 + \frac{1}{2\pi i n} - \frac{1}{2\pi i n} + \frac{1}{2\pi^2 n^2} = \frac{1}{2\pi^2 n^2}.$$

Thus,

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n\theta} = \frac{1}{12} + \sum_{\substack{x=-\infty \\ x\neq 0}}^{\infty} \frac{e^{2\pi i n\theta}}{2\pi^2 n^2}.$$

Also see (b) for an equivalent version using a single infinite sum, with which it is easy to extend $f$ periodically to all of $\mathbb{R}$.

*Proof of 5(b).* First notice that, from above, when $n \neq 0$ we have

$$\hat{f}(n)e^{2\pi i n\theta} + \hat{f}(-n)e^{-2\pi i n\theta} = \frac{1}{2\pi^2 n^2}(e^{2\pi i n\theta} + e^{-2\pi i n\theta}) = \frac{\cos(2\pi n\theta)}{\pi^2 n^2}.$$

Therefore, an alternate expression to the doubly infinite sum is

$$f(\theta) = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{\cos(2\pi n\theta)}{\pi^2 n^2}.$$

Since $\cos(2\pi n\theta)$ is periodic on $[0,1]$, we can easily extend $f$ to all of $\mathbb{R}$ without much modification.

Let $e_n(x) := e^{-2\pi i n x}$. It has been shown previously that $\|e_k\| = 1$. Since

$$\left\| \frac{1}{12} + \sum_{k=1}^{n} \frac{1}{2\pi^2 k^2} e_k \right\| \leqslant \frac{1}{12} + \sum_{k=1}^{n} \|e_k/(2\pi^2 k^2)\| = \frac{1}{12} + \sum_{k=1}^{n} \|e_k\| \left| \frac{1}{2\pi^2 k^2} \right| = \frac{1}{12} + \sum_{1}^{n} \left| \frac{1}{2\pi^2 k^2} \right|$$

and the RHS clearly converges by integral test against $f(x) = 1/(2\pi^2 x^2)$, the *Weierstraß M-test* tells us that the LHS converges absolutely uniformly to some function. In another homework problem we have shown that the partial Fourier sums of $f$ (i.e., the LHS) converge to $f$ in $\|\cdot\|_{L^2}$. Since limits are unique, it must be the case that the LHS converge uniformly to $f$. □

*Proof of 5(c).* If we consider the Fourier series at $f(0)$, we get

$$\frac{1}{4} = f(0) = \frac{1}{12} + \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2} \implies \sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2 \left[ \frac{1}{4} - \frac{1}{12} \right] = \frac{\pi^2}{6}, \text{ as desired.}$$ □