Math 425b, Spring 2021 **Name:** ____________________
Midterm Exam 2
4/2–4/6

This exam contains 16 pages (including this cover page) and 5 problems. One way to take the exam is to print it out, take it by hand as usual, and then scan and upload the result into Gradescope. Other methods are allowed, e.g. you can write on a separate sheet / sheets of paper or a tablet, or use LaTeX as long as you adhere to the time limits. In any case, make sure your submission includes your name to avoid any Gradescope issues.

The time limit for the exam is: 2 time blocks (first block 3 hours, second block 4 hours), with a break of any length in between (e.g. if you have questions about any of the problems, you can email me and wait to take your second time window until I reply, but please make sure you're completely done with the exam by Tuesday night). You are not allowed to consult any course- or exam-related resources during the break (books, notes, etc.).

You are required to write in complete sentences on this exam, with the usual exceptions for abbreviations, symbols, etc. permitted as on the homework.

You may *not* use books, notes, calculator, the Internet, or other outside resources on this exam. You are expected to conduct yourself with academic integrity in all aspects of this exam; please let me know if any concerns arise.

1. (30 points) Prove **one** of the following theorems (your choice). You may assume any lemmas that you need; only the main proof of the theorem is required. Extended hints for each can be found below.

   - Arzelá–Ascoli theorem: if X is a compact metric space and $(f_n)_{n=1}^\infty \subset C^0(X,\mathbb{R})$ is a uniformly bounded and equicontinuous set of functions, then $(f_n)_{n=1}^\infty$ has a uniformly convergent subsequence.
   - Weierstrass approximation theorem: any $f \in C^0([0,1],\mathbb{R})$ is the uniform limit of a sequence of polynomial functions.
   - Stone–Weierstrass theorem over $\mathbb{R}$: if X is a compact metric space and $\mathcal{A} \subset C^0(X,\mathbb{R})$ is a function algebra that vanishes nowhere and separates points, then $\mathcal{A}$ is dense in $C^0(X,\mathbb{R})$.
   - Picard–Lindelöf theorem (autonomous case): if $U \subset \mathbb{R}^m$ is open with $y_0 \in U$ and $F : U \to \mathbb{R}^m$ is locally Lipschitz, then there exists an open interval (a,b) containing zero and a differentiable function $\gamma : (a,b) \to U$ with $\gamma'(t) = F(\gamma(t))$ for all $t \in (a,b)$ as well as $\gamma(0) = y_0$. Any two such differentiable functions $\gamma, \tilde{\gamma}$ agree on some open neighborhood of zero.

   Hint for Arzelá–Ascoli: First pick a countable dense subset A of X (what assumption ensures that such a set A exists?). The strategy is to show $(f_n)_{n=1}^\infty$ has a subsequence converging pointwise on A, then to use the "propagation theorem" to upgrade to uniform convergence of this subsequence on all of X.

   To get a subsequence converging pointwise on $A = \{a_1, a_2, \ldots\}$, first try to get a subsequence converging at the point a_1. Then, passing to a further subsequence, try to get convergence at both a_1 and a_2, and proceed recursively in this manner. To pass from convergence on finite subsets of A to convergence on the whole infinite set A, it may help to think about Cantor's "diagonalization" proof of the uncountability of $\mathbb{R}$. To complete the proof of Arzelá–Ascoli, use the propagation theorem: are the hypotheses satisfied?

   Hint for Weierstrass approximation: For a given $n \geq 1$ and for $0 \leq k \leq n$, let r_k be the Bernstein basis polynomial

   $$r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

   Define

   $$p_n(x) = \sum_{k=0}^{n} f(k/n) r_k(x);$$

   you want to show that the polynomial functions p_n converge uniformly to f on $[0,1]$. To do this, let $\varepsilon > 0$ be given. Pick a δ that works for $\varepsilon/2$ in the definition of uniform continuity for f (why is f uniformly continuous?). Also pick M such that $|f(x)| \leq M$ for all $x \in [0,1]$ (why does M exist?). Let N be any integer $\geq \frac{M}{\varepsilon \delta^2}$ and let $n \geq N$; your goal is to show $|p_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$.

   To understand $|p_n(x) - f(x)|$ for a fixed $x \in [0,1]$, use the fact that $\sum_{k=0}^n r_k(x) = 1$ to write $f(x) = \sum_{k=0}^n f(x) r_k(x)$, so that

   $$|p_n(x) - f(x)| = \left| \sum_{k=0}^{n} (f(k/n) - f(x)) r_k(x) \right|.$$

Split the sum over k into two parts: let K_1 be the set of k such that k/n is closer than δ to x, and let K_2 be the set of k such that k/n is at least distance δ from x. Try to show that both the sum over K_1 and the sum over K_2 have absolute value $< \varepsilon/2$. For the sum over K_1, your uniform continuity assumption above should help. For the sum over K_2, use a coarse bound on $|f(k/n) - f(x)|$ coming from your bound M chosen above, and try to show that

$$\sum_{k \in K_2} (n\delta)^2 r_k(x) \leq \sum_{k \in K_2} (k - nx)^2 r_k(x).$$

Now use that the variance of the binomial distribution is $nx(1-x)$; use basic calculus to find an upper bound for $x(1-x)$ on $[0,1]$, and use $n \geq N$ to help finish the proof.

Hint for Stone–Weierstrass: Given $F \in C^0(X, \mathbb{R})$ and $\varepsilon > 0$, it suffices (why?) to find $G \in \overline{\mathcal{A}}$ with $|F(x) - G(x)| < \varepsilon$ for all $x \in X$, i.e.

$$F(x) - \varepsilon < G(x) < F(x) + \varepsilon$$

for all $x \in X$. The goal will be to find $G_p \in \overline{\mathcal{A}}$ (for each $p \in X$) with $F(x) - \varepsilon < G_p(x)$ for all $x \in X$ and $G_p(x) < F(x) + \varepsilon$ for all x in some open neighborhood V_p of p. If you can do this, then show that there exist $p_1, \ldots, p_k$ with $X = V_{p_1} \cup \cdots \cup V_{p_k}$, and take $G = \min(G_{p_1}, \ldots, G_{p_k})$ (show this satisfies the correct inequalities; you can cite a lemma to say it's in $\overline{\mathcal{A}}$).

Working backwards a bit, to find G_p, you want to apply a similar idea: assume you can find $H_{p,q} \in \overline{\mathcal{A}}$ (for each pair of points $(p,q) \in X$) such that

$$F(x) - \varepsilon < H_{p,q}(x)$$

for all x in an open neighborhood $U_{p,q}$ of q, and such that

$$H_{p,q}(x) < F(x) + \varepsilon$$

for all x in an open neighborhood $V_{p,q}$ of p. For a given p, show that there exist $q_1, \ldots, q_m$ with $X = U_{q_1} \cup \cdots \cup U_{q_m}$, and take $G_p = \max(H_{p,q_1}, \ldots, H_{p,q_m})$ (show this works).

Finally, to find $H_{p,q}$ for every (p,q), find $H_{p,q} \in \mathcal{A}$ with $H_{p,q}(p) = F(p)$ and $H_{p,q}(q) = F(q)$ (by citing a lemma), then use a continuity argument to show that open neighborhoods $q \in U_{p,q}$, $p \in V_{p,q}$ exist as described above.

Hint for Picard–Lindelöf: You can assume without loss of generality that F has a Lipschitz constant L on all of U. Pick r such that the closed ball $N := \overline{B_r(y_0)}$ is contained in U. Choose M such that $\|F(x)\| \leq M$ for all $x \in N$ (why is this possible?). Let τ be any number with $0 < \tau < \min(r/M, 1/L)$. Try to set things up so you can use the contraction mapping theorem to show there's a unique solution γ mapping $[-\tau, \tau]$ to N with initial value $\gamma(0) = y_0$.

To do this, rephrase the initial value problem $\begin{cases} \gamma'(t) = F(\gamma(t)) \\ \gamma(0) = y_0 \end{cases}$ as an integral equation. Introduce a mapping Φ from $C^0([-\tau, \tau], N)$ to itself, such that γ solves the integral equation if and only if it's a fixed point of Φ. Show that Φ actually maps $C^0([-\tau, \tau], N)$ into itself (the main issue is showing that $(\Phi(\gamma))(t)$ is in N for all $t \in [-\tau, \tau]$; the bound $\|F(x)\| \leq M$ for $x \in N$ will be useful). Show that Φ is a contraction with respect to the uniform metric $d_{\sup}$ on

$C^0([-\tau, \tau], N)$; you can use τL, which is less than 1 by assumption, as the contraction constant. Apply the contraction mapping theorem (why is it valid?).

At this point, you've shown existence (a solution $\gamma : [-\tau, \tau] \to N$ is, in particular, a solution $\gamma : (-\tau, \tau) \to U$). To show uniqueness, given γ and $\tilde{\gamma}$ as in the statement, show there exists some $0 < \tau' \leq \tau$ such that γ and $\tilde{\gamma}$ are defined on $[-\tau', \tau']$ and map $[-\tau', \tau']$ into N. Use uniqueness in the contraction mapping theorem to finish the proof.

(more space for previous problem)

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2. (a) (6 points) Let (X, d) be a metric space. Define what it means for $f : X \to X$ to be a contraction.

   (b) (16 points) State and prove the Banach contraction mapping theorem.

   Hint: For the proof, first show uniqueness: how can two points simultaneously stay fixed and move closer together? Next, show existence by studying the iterates $f^n(x)$ of any point x. Show that these iterates form a Cauchy sequence. To do this, use an estimate based on the partial sums of the geometric series $\sum k^n$ where k is the contraction constant. Why does the sequence of iterates converge, and what more can you say about its limit?

(More space for previous problem)

3. (22 points) Do **one** of the following problems (your choice).

- Define what it means for a subset $A \subset X$ of a metric space X to be dense. Prove that if X is compact and $A = \{a_1, a_2, \ldots\}$ is a countable dense subset of X and $\delta > 0$, then there exists N such that for all $x \in X$, there exists $i \in \{1, \ldots, N\}$ with $d(x, a_i) < \delta$ (the "δ-density lemma").

  Hint: Try to apply Dini's theorem, where the decreasing sequence of continuous functions $f_n : X \to \mathbb{R}$ is defined with the help of the distance function on X. Use density of A to get pointwise convergence of this sequence of functions to the zero function, and use the conclusion of Dini's theorem to finish the proof.

- Let X be a metric space; define what it means for a subset $\mathcal{A} \subset C^0(X, \mathbb{R})$ to be a function algebra. Prove that if X is compact, $\mathcal{A} \subset C^0(X, \mathbb{R})$ is a function algebra, and $f \in \mathcal{A}$, then the function $|f|$ defined by $|f|(x) := |f(x)|$ is in the closure $\overline{\mathcal{A}}$.

  Hint: Writing $|f|$ as the composition of $f : X \to \mathbb{R}$ with the absolute value function from $\mathbb{R}$ to $\mathbb{R}$, the goal is to replace the absolute value function by a polynomial function $q(x)$ with no constant term (i.e. with $q(0) = 0$) that is uniformly close to the absolute value function. For such a function q, you should be able to show that the composition $q \circ f$ is in $\mathcal{A}$, and also that $q \circ f$ is uniformly close to $|f|$.

  To define q, you can apply the Weierstrass approximation theorem; on which compact interval are you applying it? Make sure you "fix" the constant term of q.

(more space for previous problem)

4. (22 points) Do **one** of the following problems (your choice).

- Let $f : \mathbb{R} \to \mathbb{C}$ be a "suitably nice" function. Write down the integral defining the Fourier transform $\hat{f}(\xi)$ for $\xi \in \mathbb{R}$. Using Fourier transforms, find one solution $f(t)$ to the differential equation

$$f'(t) + 2f(t) = \cos(2\pi t)$$

  and check explicitly that your solution is valid.

  Hint: Write $\cos(2\pi t)$ in terms of exponential functions $e^{2\pi int}$; compute the Fourier transforms of the resulting exponentials, which should be Dirac delta functions. Using properties of the Fourier transform, you can get a formula for $\hat{f}(\xi)$. Plug this into the Fourier inversion formula to get a formula for $f(t)$; the presence of the δ function should get rid of the integral. Then verify (by computing derivatives explicitly) that your solution is valid; it may help to simplify the solution first.

- Let $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$, the set of continuous 1-periodic functions from $\mathbb{R}$ to $\mathbb{C}$ (so that $f(x+n) = f(x)$ for all integers n and real numbers x). Define the Fourier coefficients and the Fourier series of f. Prove that the Fourier series of f converges to f in the L^2 norm

$$\|f\|_2 := \sqrt{\int_0^1 |f(x)|^2 dx}.$$

  Hint: Given ε, using the Stone–Weierstrass theorem, find an element p of

$$\mathcal{A} := \text{span}\{\cdots, e_{-2}, e_{-1}, e_0, e_1, e_2, \cdots\}$$

  (where $e_n(x) = e^{2\pi inx}$) that is close enough to f in the norm $\|\cdot\|_{\text{sup}}$ (make sure you check the conditions of the theorem, and figure out "close enough" at the end of the proof). Since p is a finite linear combination of the functions e_n, there exist N, M such that p is in $\text{span}\{e_{-M}, e_{-M+1}, \cdots, e_{N-1}, e_N\}$, which itself is contained in

$$\text{span}\{e_{-m}, e_{-m+1}, \cdots, e_{n-1}, e_n\}$$

  for any $n \geq N$, $m \geq M$. For such n and m, try to bound the L^2 norm of $f - \sum_{k=-m}^{n} \hat{f}(k)e_k$ (recall what the partial sums of the Fourier series mean in terms of orthogonal projections onto subspaces, which give "L^2-closest elements" of a subspace to some point), and fix "close enough" earlier in the proof accordingly.

(More space for previous problem)

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5. Define $f \in C^0_{\text{per}}(\mathbb{R}, \mathbb{C})$ by setting $f(x) = (1/2 - x)^2$ for $x \in [0, 1]$ and extending f periodically to all of $\mathbb{R}$ (since f is continuous on $[0, 1]$ and $f(0) = f(1) = 1/4$, the periodic extension of f is continuous on all of $\mathbb{R}$).

   (a) (9 points) Compute the Fourier coefficients $\hat{f}(n)$ for all $n \in \mathbb{Z}$, and write down the corresponding Fourier series as a doubly infinite sum of exponentials.

   Hint: Computing the Fourier coefficients is an integration-by-parts problem. Expand out $f(x)$ as $\frac{1}{4} - x + x^2$. You can compute the $n = 0$ Fourier coefficient directly, but for $n \neq 0$, it might help to compute the Fourier coefficients of each of these three functions individually. Start with the constant term (what is $e^{2\pi i n}$ equal to, for any integer n?), then do the linear term, then finally the quadratic term (the computation for the linear term will help in the quadratic term).

(b) (9 points) Prove that the Fourier series you computed above converges uniformly to f.

Hint: First show that the Fourier series you computed converges uniformly to something (there are many ways to say this; you could try the Weierstrass M-test, for example, or just argue directly). Now use L^2 convergence of Fourier series to deduce that the uniform limit of the Fourier series (which must also be the L^2 limit) is actually f (can you find a way to use uniqueness of limits? It's useful that the given function f is continuous, since $f(0) = f(1)$).

(c) (4 points) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (this is known as the Basel problem, first solved by Euler in 1734).

Hint: The identity should follow from taking $x = 0$ in your Fourier series above; this might be a good way to check the constants in your computation.

(More space for previous problem or scratch work)