Stone-Weierstraß Theorem (Stone, 1937)

Before proving the Stone-Weierstraß Theorem, we need some lemmas first.

Lemma 0.0.0.1

If $\mathcal{A} \in C^0(X, \mathbb{R})$ is a function algebra, then so is its closure $\overline{\mathcal{A}}$ (i.e., closed).

Proof. $\overline{\mathcal{A}}$ is closed under sums: if $f, g \in \overline{\mathcal{A}}$ such that there exist $\{f_n\}, \{g_n\} \subset \mathcal{A}$ with $f_n \to f$ and $g_n \to g$ uniformly, then $f_n + g_n \to f + g$ uniformly. Similarly, $\overline{\mathcal{A}}$ is closed under scalar multiples and function multiples. (Proofs are not hard.) $\square$

Lemma 0.0.0.2

Approximation of continuous functions by polynomials no only work for $f \in C^0([0,1], \mathbb{R})$ but also for $f \in C^0([a,b], \mathbb{R})$. *We'll focus on $f(x) = |x|$ on $[-1,1]$ later, thanks to this lemma.*

Proof. We simply need to linearly rescale the functions by $\Phi : C^0([a,b], \mathbb{R}) \to C^0([0,1], \mathbb{R})$ with

$$f(x) \mapsto f((x-a)/(b-a)) \text{ and } f^{-1}(y) =\mapsto f^{-1}(a + y(b-a))$$

(where $x \in [a,b]$ and $y \in [0,1]$). Notice that these operations are linear transformations of vector spaces (isomorphism, in particular). Thus they preserve $\| \cdot \|_{\sup}$.

Therefore, $f_n \to f$ uniformly in $C^0([a,b], \mathbb{R})$ if and only if the corresponding $\Phi(f_n) \in C^0([0,1], \mathbb{R})$ converge uniformly to $\Phi(f) \in C^0([0,1], \mathbb{R})$.

Furthermore, these linear transformations preserve the subset of polynomial functions, i.e., polynomials in $C^0([a,b], \mathbb{R})$ gets mapped to polynomials in $C^0([0,1], \mathbb{R})$ and vice versa. This proves the lemma. $\square$

Now two slightly more difficult lemmas.

Lemma 0.0.0.3

Let X be compact and let $\mathcal{A} \subset C^0(X, \mathbb{R})$ be a function algebra (and $\overline{\mathcal{A}}$ is also a function algebra). Then if $f \in \overline{\mathcal{A}}$, $|f| \in \overline{\mathcal{A}}$, i.e., $\overline{\mathcal{A}}$ also has closure under absolute values.

Proof. Main idea: $|f|$ is a composition of f and the usual absolute function: $X \xrightarrow{f} \mathbb{R} \xrightarrow{|\cdot|} \mathbb{R}$. We will approximate the second absolute value function by a polynomial (on some compact interval containing the image of f as Weierstraß approximation works for compact domain). Ideally, we want to approximate $|x|$ by $a_1 x + a_2 x^2 + \cdots + a_n x^n$. Then,

$$(|\cdot| \circ f)(x) = a_1 f + a_2 f^2 + \cdots + a_n f^n,$$

where each f^k is obtained from function multiplication and are in $\overline{\mathcal{A}}$, and $|f|$ is a linear combination of elements of $\overline{\mathcal{A}}$. Thus $f \in \overline{\overline{A}} = \overline{A}$.

<u>Formally</u>: note that $\|f\|_{\sup}$ is finite since X is compact. Consider the absolute value function

$$[-\|f\|_{\sup}, \|f\|_{\sup}] \to \mathbb{R} \text{ defined by } y \mapsto |y|.$$

This is continuous, and by the lemma involving "transformations" of Weierstrß approximation there exists a polynomial $p(y)$ such that $|p(y) - |y|| < \epsilon/2$ for all y in the domain. In parcicular, $|p(0) - |0|| < \epsilon/2$, i.e., $|p(0)| < \epsilon/2$. Therefore, there exists a polynomial $q(y)$ such that $q(0) = 0$ (<u>i.e., the constant term vanishes!!</u>) and $|q(y) - |y|| < \epsilon$. To see this, simply let $q(y) := p(y) - p(0)$. We know $|p(0) - |0|| < \epsilon/2$. Therefore for all y in the domain,

$$|q(y) - |y|| = |p(y) - p(0) - |y|| \leqslant |p(y) - |y|| + |p(0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now we expand q as $q(y) = a_1 y + a_2 y^2 + \cdots + a_n y^n$, <u>without constant term</u>. Let $g : X \to \mathbb{R}$ be defined by $g = q \circ f$. Then

$$g = a_1 f + a_2 f^2 + \cdots + a_n f^n \in \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}.$$

We want to show that g is a nice approximation of $|f|$, i.e., $\|g - |f|\|_{\sup}$ is small. Indeed,

$$\|g - |f|\|_{\sup} = \sup_{x \in X} |q(f(x)) - |f(x)||$$

$$= \sup_{y \in \mathrm{im}(f)} |q(y) - |y|| \leqslant \epsilon$$

since the image of f is a subset of $[-\|f\|_{\sup}, \|f\|_{\sup}]$. Therefore $|f|$ can be approximated arbitrarily closely by elements of $\overline{\mathcal{A}}$. Thus $|f| \in \overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}$.

$\square$

Immediately following the above lemma, we have the following:

Lemma 0.0.0.4

Let X be compact and let $\mathcal{A} \subset C^0(X, \mathbb{R})$ be a function algebra. If $f, g \in \overline{\mathcal{A}}$m then $\max(f, g), \min(f, g)$ (<u>pointwise</u> maximum / minimum) are also in $\overline{\mathcal{A}}$.

Proof. Indeed,
$$\max(a, b) = \frac{a + b}{2} + \frac{|a - b|}{2} \text{ and } \min(a, b) = \frac{a + b}{2} - \frac{|a - b|}{2}.$$
We either get a or b when evaluating the max or min. Therefore if $f, g \in \overline{\mathcal{A}}$ then so are $\max(f, g), \min(f, g)$.

$\square$

One more lemma to go! To be shown next lecture.

Lemma 0.0.0.5

Let X be compact. Assume $\mathcal{A} \subset C^0(X, \mathbb{R})$ satisfies all hypotheses of the Stone-Weierstraß theorem (i.e., a function algebra that vanishes nowhere and separates points), then we have a stronger result: if $x_1, x_2 \in X$ with $x_1 \neq x_2$ and given $c_1, c_2 \in \mathbb{R}$, then there exists $f \in \mathcal{A}$ with $f(x_1) = c_1$ and $f(x_2) = c_2$. In other words, we can <u>prescribe</u> the values at 2 points $\in X$ and get $f \in \mathcal{A}$ which <u>satisfies this prescription</u>.