Beginning of March 22, 2021

The Picard-Lindelöf Theorem can be generalized into the following form (that can be applied to differential forms):

Theorem 0.0.1: Picard-Lindelöf Theorem, "time-dependent case"

(Picard-Lindelöf, ~ 1880; Lipschitz 1876.) Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ be open (the first $\mathbb{R}$ represent *time*). Let $F : \Omega \to \mathbb{R}^n$ a "*time-varying vector field*". Let elements of Ω be of form (t, y) where $t \in \mathbb{R}$ (time) and $y \in \mathbb{R}^n$ (spatial coordinates). Assume

(1) F is continuous on Ω, "jointly continuous in t and in y, and

(2) F is locally Lipschitz in y and locally uniformly in t, i.e., for all $(t_0, y_0) \in \Omega$, there exists an open neighborhood V of $(t_0, y_0) \in \Omega$ and $L \geqslant 0$ such that, whenever $(t, y), (t, y') \in V$, the Lipschitz condition holds. *Must compare t with the same t!*

Then for all $(t_0, y_0) \in \Omega$, there exists an open neighborhood (a, b) of t_0 and a differentiable function $\gamma : (a, b) \to \mathbb{R}^n$ such that

(1) For all $t \in (a, b)$, the pair $(t, \gamma(t)) \in \Omega$,

(2) $\gamma'(t) = F(t, \gamma(t))$ for all $t \in (a, b)$, and

(3) $\gamma(t_0) = y_0$.

Local uniqueness holds as before.

Remark. Differential equations like $f''(t) - 3'(t) + 2f(t) = 0$ is an autonomous system (now and also after reduction of order). On the other hand, something like $f''(t) = 3f'(t) + 2f(t) = \cos(t)$ is non-autonomous as now the RHS is of form $g(t)$. This corresponds to the more general time-varying case.

Contraction

Now we talk about fixed points of functions $f : X \to X$.

Definition 0.0.2

Let X be a set and $f : X \to X$ a function. We say $x \in X$ is called a **fixed point** of f if $f(x) = x$. The **orbit** of x under f is given by $\{x, f(x), f^2(x), \dots\}$. Thus equivalently x is a fixed point of f if and only if its orbit is just $\{x\}$.

Example 0.0.3: Brouwer Fixed-Point Theorem. Any continuous $f : D^n \to D^n$ has at $\geqslant 1$ fixed point.

We will be especially concerned with fixed points for contraction mappings:

Definition 0.0.4

If (X,d) is a metric and $f : X \to X$ a function, we say f is a **contraction** (or contractive mapping) if there exists $k < 1$ such that $d(f(x), f(y)) \leqslant k(x,y)$ for all $x, y \in X$. If such $k < 1$ does not exist and $d(f(x), f(y)) < d(x,y)$ for all x, y, f is called a **weak contraction**.

Example 0.0.5. Let $X := [1, \infty)$. $f : X \to X$ defined by $f(x) = x + 1/x$ is a weak contraction but not a contraction. It does not admit a fixed point as $f(x) > x$ for all x.

Theorem 0.0.6: Banach Contraction Mapping Theorem

Let (X,d) be a complete metric space and $f : X \to X$ a contraction. Then f has a unique fixed point.