Part 1 Banach and Hilbert Spaces

1.3 If Ω is bounded, what is the completion of $C_c^0(\Omega)$ in the supremum norm? Deduce that $C_c^0(\Omega)$ is not a Banach space with this norm. Treat similarly the case $\Omega = \mathbb{R}^m$.

Solution

The completion of $C_c^0(\Omega)$ is given by the space of continuous functions on $\overline{\Omega}$ that vanishes on $\partial\Omega$. To see this, on one hand any $f_n \in C_c^0(\Omega)$ has $f_n \equiv 0$ on $\partial\Omega$, so if $\|f_n - f\|_\infty \to 0$, $f \equiv 0$ on $\partial\Omega$ too. Now it remains to show that every element of our claimed completion space is indeed a limit of some sequence $\{f_n\} \subset C_c^0(\Omega)$.

Indeed, take any g from the completion. Define a sequence of functions $\{f_n\}$ by

$$g_n(x) := \operatorname{sgn} g(x) \cdot \max(0, |g(x)| - 1/n)$$

so that $g_n(x)$ and $g(x)$ never have the opposite signs, $|g_n(x)| \equiv |g(x)| - 1/n$, unless $|g(x)| < 1/n$, in which case we set $g_n(x) = 0$. It is clear that $\|g_n - g\|_\infty \leqslant 1/n$, and it's also clear that each $g_n \in C_c^0(\Omega)$ since the supp g_n is at least $1/n$ away from $\partial\Omega$.

To see that this completion space does not equal the original $C_c^0(\Omega)$, consider a continuous function h on $\overline{\Omega}$ with $h(x) > 0$ for all $x \in \Omega$. It follows that supp $h = \overline{\Omega} \not\subset\not\subset \Omega$ (*I hope this symbol doesn't look weird... it's supposed to mean "not a compact subset of"*). Therefore $C_c^0(\Omega)$ is not Banach.

The completion of $C_c^0(\mathbb{R}^m)$ is given by $\{f \in C_b^0(\mathbb{R}^m) : f(x) \to 0 \text{ as } |x| \to \infty\}$. As usual, we start by taking a sequence $\{f_n\}$ that converges (uniformly) to f. Then, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_m - f\|_\infty < \epsilon$ whenever $m > N$. On the other hand, since $f_m \in C_c^0(\Omega)$, its support is bounded, so there exists a sufficiently large k_m such that $|f_m(x)| \equiv 0$ whenever $|x| > k_m$. Therefore $|f(x)| < \epsilon$ whenever $|x| > k_m$. This shows the "$\subset$" direction of our claim.

Now we show "$\supset$", starting by taking any f in our claimed completion space. Just like above, we can construct a sequence $\{f_n\} \subset C_c^0(\mathbb{R}^m)$ with $\|f_n - f\|_\infty \leqslant 1/n$. Once again, this completion space strictly contains $C_c^0(\Omega)$: the function $g(x) := 1/|x|$ is nowhere zero but indeed tends to 0 whenever $|x| \to \infty$.

1.4 There is no norm that makes $C^\infty(\overline{\Omega})$ into a Banach space. However, there are various subspaces of $C^\infty(\overline{\Omega})$ that are Banach spaces. For example, for any sequence $c = \{c_n\}_{n\geqslant 1}$ define the norm

$$\|f\|_c := \sum_{n=1}^{\infty} c_n \|f\|_{C^n(\overline{\Omega})}.$$

Show that the subspace of $C^\infty(\overline{\Omega})$ consisting of all those f with $\|f\|_c$ finite is a Banach space.

Proof. Let a sequence$\{f_k\}_{k\geqslant 1} \subset C^\infty(\overline{\Omega})$ with finite norms be Cauchy with respect to $\|\cdot\|_c$. By non-degeneracy of norms, for each k, $\|f_k\|_{C^n(\overline{\Omega})}$ also form a Cauchy sequence. Since $C^n(\overline{\Omega})$ is complete (in fact separable, p.17) for finite n, $f_k \to f$ in $C^n(\overline{\Omega})$. It remains to show that $f_k \to f$ with respect to $\|\cdot\|_c$ and that this norm is finite.

Indeed,

$$\|f_k - f\|_c = \sum_{n=1}^{\infty} c_n \|f_k - f\|_{C^n(\overline{\Omega})} = \lim_{j\to\infty} \sum_{n=1}^{j} c_n \underbrace{\|f_k - f\|_{C^n(\overline{\Omega})}}_{\to 0} \to 0,$$

and

$$\|f\|_c \leqslant \|f - f_k\|_c + \|f_k\|_c < \epsilon + \underbrace{\|f\|_c}_{<\infty} < \infty.$$

Therefore both claims $f_k \to f$ and $\|f\|_c < \infty$ have been proven. Indeed introducing $\{c_n\}$ is a "remedy". ☐

1.5 Show that $C^{r,\gamma}(\overline{\Omega})$ is Banach.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C^{r,\gamma}(\overline{\Omega})$. It follows that the sequence is uniformly bounded; in particular, the "Hölder ratio with exponent γ" of the sequence is bounded, say by M. Observe that this sequence is equicontinuous. Indeed, given $\epsilon > 0$, letting $\delta > 0$ be small enough such that $M\delta^\gamma < \epsilon$ suggests that for all n and all $x, y \in \overline{\Omega}$ with $|x - y| < \delta$,

$$|D^\alpha f_n(x) - D^\alpha f_n(y)| \leqslant M|x - y|^\gamma < M\delta^\gamma < \epsilon.$$

Therefore, by Arzelà-Ascoli there exists a subsequence of $\{f_n\}$ that converges (uniformly) to some $f \in C^0(\overline{\Omega})$. Notice that we can argue analogously and show $D^\alpha f_n \to$ some $f_\alpha \in C^0(\overline{\Omega})$ when $|\alpha| \leqslant r$.

We first show that $D^\alpha f = f_\alpha$. Indeed, inductively, if $f_n(x) \to f(x)$ uniformly, differentiating

$$\lim_{n\to\infty} f_n(x) = f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)) = f(x)$$

with respect to any $|\alpha| = 1$ gives

$$\lim_{n\to\infty} D^\alpha f_n(x) = D^\alpha f(x) \text{ uniformly, i.e., } D^\alpha f_n \to D^\alpha f \text{ uniformly.}$$

Now it remains to show $f \in C^{r,\gamma}(\overline{\Omega})$ and that $f_n \to f$ in $C^{r,\gamma}(\overline{\Omega})$. To shown the Hölder condition, since $D^\alpha f_n \to D^\alpha f$ uniformly, for some sufficiently large N,

$$|D^\alpha f_N(\tilde{x}) - D^\alpha f(\tilde{x})| < \frac{|x - y|^\gamma}{2} \text{ for all } \tilde{x} \in \overline{\Omega}.$$

Then, using the "splitting into three parts" trick, for all $x, y \in \overline{\Omega}$,

$$\begin{aligned}
|D^\alpha f(x) - D^\alpha f(y)| &\leqslant |D^\alpha f(x) - D^\alpha f_N(x)| + |D^\alpha f_N(x) - D^\alpha f_N(y)| + |D^\alpha f_N(y) - D^\alpha f(y)| \\
&\leqslant \frac{|x - y|^\gamma}{2} + M|x - y|^\gamma + \frac{|x - y|^\gamma}{2} \\
&= \underbrace{1 + M}_{<\infty}|x - y|^\gamma,
\end{aligned}$$

and the claim follows. To see that $f_n \to f$ in $C^{r,\gamma}(\overline{\Omega})$, notice that, for any $x, y \in \overline{\Omega}$,

$$\begin{aligned}
&\|(D^\alpha f_n(x) - D^\alpha f_n(y)) - (D^\alpha f(x) - D^\alpha f(y))\|_{C^{r,\gamma}} \\
&\leqslant \limsup_{m\to\infty} \|(D^\alpha f_n(x) - D^\alpha f_m(x)) - (D^\alpha f_n(y) - D^\alpha f_m(y))\| \\
&\leqslant C|x - y|^\gamma
\end{aligned}$$

where C is the "Hölder constant" of $D^\alpha f_n - D^\alpha f_m$, or alternatively $\|f_n - f_m\|_{C^{r,\gamma}} - \|f_n - f_m\|_{C^r}$." Since $\{f_n\}$ is Cauchy, this constant must converge to 0 as $m, n \to \infty$. Therefore $f_n \to f$ and we are done. ☐