1.6 Show that if Ω is convex and bounded, then any $C^1(\overline{\Omega})$ function is Lipschitz.

> ***Proof.*** By assumption, if $f \in C^1(\overline{\Omega})$ then its derivative is bounded, i.e., $|Df(x)| < M$ uniformly for some M. Pick $x, y \in \Omega$. Since Ω is convex, $y + (1-\lambda)(x-y) \in \Omega$ for all $\lambda \in [0,1]$. Therefore we can use the trick
>
> $$\begin{aligned} f(x) - f(y) &= \left| \int_0^1 Df(y + \tilde{\lambda}(x-y)) \underbrace{(x-y)}_{\text{chain rule}} \, \mathrm{d}\tilde{\lambda} \right| \\ &\leqslant \left| \int_0^1 Df(y + \tilde{\lambda}(y-x) \, \mathrm{d}\tilde{\lambda} \right| |x-y| \\ &< L|x-y| \end{aligned}$$
>
> and the claim follows. □

1.8 Prove the generalized Hölder inequality: if $\sum_{i=1}^{n} \frac{1}{p_i} = 1$ and $f_i \in L^{p_k}(\Omega)$ then the product $f_1 \dots f_n \in L^1(\Omega)$, with

$$\int_\Omega |f_1(x) \dots f_n(x)| \, \mathrm{d}x \leqslant \|f_1\|_{L^{p_1}} \dots \|f_n\|_{L^{p_n}}.$$

> ***Proof.*** The base case $n = 2$ is already given by Hölder inequality. For the inductive step, assume the inequality holds for any k numbers. Now let $p_1, \dots, p_{k+1}$ be given with sum of reciprocal 1. If we define p to be such that $1/p = 1/p_k + 1/p_{k+1}$, then $1 = p/p_k + p/p_{k+1}$ and applying the normal Hölder inequality gives
>
> $$\|f_k f_{k+1}\|_{L^p}^p = \|f_k^p f_{k+1}^p\|_{L^1} \overset{(H)}{\leqslant} \|f^p\|_{L^{p_k/p}} \|f^p\|_{L^{p_{k+1}/p}} = \|f_k\|_{L^{p_k}}^p \|f_{k+1}\|_{L^{p_{k+1}}}^p,$$
>
> so $\|f_k f_{k+1}\|_{L^p} \leqslant \|f_k\|_{L^{p_k}} \|f_{k+1}\|_{L^{p_{k+1}}}$. Using our induction hypothesis with $\sum_{i=1}^{k-1} \frac{1}{p_i} + \frac{1}{p} = 1$,
>
> $$\begin{aligned} \int_\Omega |f_1(x) \dots f_{k+1}(x)| \mathrm{d}x &\leqslant \|f_1\|_{L^1} \dots \|f_{k-1}\|_{L^{p_{k-1}}} \|f_k f_{k+1}\|_{L^p} \\ &\leqslant \|f_1\|_{L^1} \dots \|f_{k-1}\|_{L^{p_{k-1}}} \|f_k\|_{L^{p_k}} \|f_{k+1}\|_{L^{p_{k+1}}}, \end{aligned}$$
>
> which finishes the proof. □

1.9 Use Hölder's inequality to obtain the L^p interpolation inequality,

$$\|u\|_{L^p} \leqslant \|u\|_{L^q}^{q(r-p)/p(r-q)} \|u\|_{L^r}^{r(p-q)/q(r-q)},$$

where $q < p < r$ and $u \in L^r(\Omega)$.

> ***Proof.*** Notice that $\frac{r-p}{r-q} + \frac{p-q}{r-q} = 1$. It follows that
>
> $$\begin{aligned} p = r - (r-p) &= r - (r-q)\frac{r-p}{r-q} \\ &= q \cdot \frac{r-p}{r-q} + r - r \cdot \frac{r-p}{r-q} \\ &= q \cdot \frac{r-p}{r-q} + r \cdot \frac{p-q}{r-q}, \end{aligned}$$
>
> so
>
> $$|u|^p = (|u|^q)^{(r-p)/(r-q)} (|u|^r)^{(p-q)/(r-q)}.$$

Then by Hölder inequality we have

$$\|u\|_{L^p}^p = \int_\Omega |u(\tilde{x})|^p \, \mathrm{d}\tilde{x} \leqslant \left(\int_\Omega |u(\tilde{x})|^q \, \mathrm{d}\tilde{x}\right)^{(r-p)/(r-q)} \left(\int_\Omega |u(\tilde{x})|^r \, \mathrm{d}\tilde{x}\right)^{(p-q)/(r-q)}$$
$$= \|u\|_{L^q}^{q(r-p)/(r-q)} \|u\|_{L^r}^{r(p-q)/(r-q)}.$$

Taking p^{th} root of both sides gives us the L^p interpolation inequality. □

1.10 Show that given $s \in S(\Omega)$, $p \in [1, \infty)$, and $\epsilon > 0$, there exists an $f \in C_c^0(\Omega)$ such that

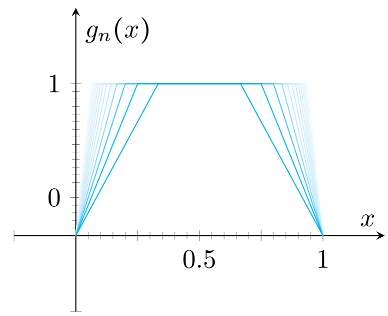
$$\|f - s\|_{L^p} < \epsilon.$$

Proof. Assume $\Omega \subset \mathbb{R}^m$. By definition $s \in S(\Omega)$ is of form

$$s(x) = \sum_{i=1}^n c_i \chi[I_i](x) \text{ where } I_i = \prod_{k=1}^m [a_{i,k}, b_{i,k}], \text{ i.e., boxes in } \mathbb{R}^m.$$

For each $[a_{i,k}, b_{i,k}]$, consider the following approximation:

$$g_n(x) = \begin{cases} n(x-a), & x \in [a, a+1/n], \\ 1, & x \in (a+1/n, b-1/n), \\ n(b-x) & x \in [b-1/n, b]. \end{cases}$$



□