Chapter 1 Exercises

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1.3 If Ω is bounded, what is the completion of $C_c^0(\Omega)$ in the supremum norm? Deduce that $C_c^0(\Omega)$ is not a Banach space with this norm. Treat similarly the case $\Omega = \mathbb{R}^m$.

Solution

The completion of $C_c^0(\Omega)$ is given by the space of continuous functions on $\overline{\Omega}$ that vanishes on $\partial\Omega$. To see this, on one hand any $f_n \in C_c^0(\Omega)$ has $f_n \equiv 0$ on $\partial\Omega$, so if $\|f_n - f\|_\infty \to 0$, $f \equiv 0$ on $\partial\Omega$ too. Now it remains to show that every element of our claimed completion space is indeed a limit of some sequence $\{f_n\} \subset C_c^0(\Omega)$.

Indeed, take any g from the completion. Define a sequence of functions $\{f_n\}$ by

$$g_n(x) := \operatorname{sgn} g(x) \cdot \max(0, |g(x)| - 1/n)$$

so that $g_n(x)$ and $g(x)$ never have the opposite signs, $|g_n(x)| \equiv |g(x)| - 1/n$, unless $|g(x)| < 1/n$, in which case we set $g_n(x) = 0$. It is clear that $\|g_n - g\|_\infty \leqslant 1/n$, and it's also clear that each $g_n \in C_c^0(\Omega)$ since the supp g_n is at least $1/n$ away from $\partial\Omega$.

To see that this completion space does not equal the original $C_c^0(\Omega)$, consider a continuous function h on $\overline{\Omega}$ with $h(x) > 0$ for all $x \in \Omega$. It follows that supp $h = \overline{\Omega} \not\subset\not\subset \Omega$ (*I hope this symbol doesn't look weird... it's supposed to mean "not a compact subset of"*). Therefore $C_c^0(\Omega)$ is not Banach.

The completion of $C_c^0(\mathbb{R}^m)$ is given by $\{f \in C_b^0(\mathbb{R}^m) : f(x) \to 0 \text{ as } |x| \to \infty\}$. As usual, we start by taking a sequence $\{f_n\}$ that converges (uniformly) to f. Then, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_m - f\|_\infty < \epsilon$ whenever $m > N$. On the other hand, since $f_m \in C_c^0(\Omega)$, its support is bounded, so there exists a sufficiently large k_m such that $|f_m(x)| \equiv 0$ whenever $|x| > k_m$. Therefore $|f(x)| < \epsilon$ whenever $|x| > k_m$. This shows the "$\subset$" direction of our claim.

Now we show "$\supset$", starting by taking any f in our claimed completion space. Just like above, we can construct a sequence $\{f_n\} \subset C_c^0(\mathbb{R}^m)$ with $\|f_n - f\|_\infty \leqslant 1/n$. Once again, this completion space strictly contains $C_c^0(\Omega)$: the function $g(x) := 1/|x|$ is nowhere zero but indeed tends to 0 whenever $|x| \to \infty$.

1.4 There is no norm that makes $C^\infty(\overline{\Omega})$ into a Banach space. However, there are various subspaces of $C^\infty(\overline{\Omega})$ that are Banach spaces. For example, for any sequence $c = \{c_n\}_{n\geqslant 1}$ define the norm

$$\|f\|_c := \sum_{n=1}^{\infty} c_n \|f\|_{C^n(\overline{\Omega})}.$$

Show that the subspace of $C^\infty(\overline{\Omega})$ consisting of all those f with $\|f\|_c$ finite is a Banach space.

Proof. Let a sequence $\{f_k\}_{k\geqslant 1} \subset C^\infty(\overline{\Omega})$ with finite norms be Cauchy with respect to $\|\cdot\|_c$. By non-degeneracy of norms, for each k, $\|f_k\|_{C^n(\overline{\Omega})}$ also form a Cauchy sequence. Since $C^n(\overline{\Omega})$ is complete (in fact separable, p.17) for finite n, $f_k \to f$ in $C^n(\overline{\Omega})$. It remains to show that $f_k \to f$ with respect to $\|\cdot\|_c$ and that this norm is finite.
Indeed,

$$\|f_k - f\|_c = \sum_{n=1}^{\infty} c_n \|f_k - f\|_{C^n(\overline{\Omega})} = \lim_{j\to\infty} \sum_{n=1}^{j} c_n \underbrace{\|f_k - f\|_{C^n(\overline{\Omega})}}_{\to 0} \to 0,$$

and

$$\|f\|_c \leqslant \|f - f_k\|_c + \|f_k\|_c < \epsilon + \underbrace{\|f\|_c}_{<\infty} < \infty.$$

Therefore both claims $f_k \to f$ and $\|f\|_c < \infty$ have been proven. Indeed introducing $\{c_n\}$ is a "remedy". □

1.5 Show that $C^{r,\gamma}(\overline{\Omega})$ is Banach.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $C^{r,\gamma}(\overline{\Omega})$. It follows that the sequence is uniformly bounded; in particular, the "Hölder ratio with exponent γ" of the sequence is bounded, say by M. Observe that this sequence is equicontinuous. Indeed, given $\epsilon > 0$, letting $\delta > 0$ be small enough such that $M\delta^\gamma < \epsilon$ suggests that for all n and all $x, y \in \overline{\Omega}$ with $|x - y| < \delta$,

$$|D^\alpha f_n(x) - D^\alpha f_n(y)| \leqslant M|x - y|^\gamma < M\delta^\gamma < \epsilon.$$

Therefore, by Arzelà-Ascoli there exists a subsequence of $\{f_n\}$ that converges (uniformly) to some $f \in C^0(\overline{\Omega})$. Notice that we can argue analogously and show $D^\alpha f_n \to$ some $f_\alpha \in C^0(\overline{\Omega})$ when $|\alpha| \leqslant r$.

We first show that $D^\alpha f = f_\alpha$. Indeed, inductively, if $f_n(x) \to f(x)$ uniformly, differentiating

$$\lim_{n\to\infty} f_n(x) = f_1(x) + \sum_{k=1}^{\infty} (f_{k+1}(x) - f_k(x)) = f(x)$$

with respect to any $|\alpha| = 1$ gives

$$\lim_{n\to\infty} D^\alpha f_n(x) = D^\alpha f(x) \text{ uniformly, i.e., } D^\alpha f_n \to D^\alpha f \text{ uniformly.}$$

Now it remains to show $f \in C^{r,\gamma}(\overline{\Omega})$ and that $f_n \to f$ in $C^{r,\gamma}(\overline{\Omega})$. To shown the Hölder condition, since $D^\alpha f_n \to D^\alpha f$ uniformly, for some sufficiently large N,

$$|D^\alpha f_N(\tilde{x}) - D^\alpha f(\tilde{x})| < \frac{|x - y|^\gamma}{2} \text{ for all } \tilde{x} \in \overline{\Omega}.$$

Then, using the "splitting into three parts" trick, for all $x, y \in \overline{\Omega}$,

$$\begin{aligned}|D^\alpha f(x) - D^\alpha f(y)| &\leqslant |D^\alpha f(x) - D^\alpha f_N(x)| + |D^\alpha f_N(x) - D^\alpha f_N(y)| + |D^\alpha f_N(y) - D^\alpha f(y)| \\ &\leqslant \frac{|x-y|^\gamma}{2} + M|x-y|^\gamma + \frac{|x-y|^\gamma}{2} \\ &= \underbrace{1+M}_{<\infty}|x-y|^\gamma,\end{aligned}$$

and the claim follows. To see that $f_n \to f$ in $C^{r,\gamma}(\overline{\Omega})$, notice that, for any $x, y \in \overline{\Omega}$,

$$\begin{aligned}&\|(D^\alpha f_n(x) - D^\alpha f_n(y)) - (D^\alpha f(x) - D^\alpha f(y))\|_{C^{r,\gamma}} \\ &\leqslant \limsup_{m\to\infty} \|(D^\alpha f_n(x) - D^\alpha f_m(x)) - (D^\alpha f_n(y) - D^\alpha f_m(y))\| \\ &\leqslant C|x-y|^\gamma\end{aligned}$$

where C is the "Hölder constant" of $D^\alpha f_n - D^\alpha f_m$, or alternatively $\|f_n - f_m\|_{C^{r,\gamma}} - \|f_n - f_m\|_{C^r}$." Since $\{f_n\}$ is Cauchy, this constant must converge to 0 as $m, n \to \infty$. Therefore $f_n \to f$ and we are done. □

1.6 Show that if Ω is convex and bounded, then any $C^1(\overline{\Omega})$ function is Lipschitz.

Proof. By assumption, if $f \in C^1(\overline{\Omega})$ then its derivative is bounded, i.e., $|Df(x)| < M$ uniformly for some M. Pick $x, y \in \Omega$. Since Ω is convex, $y + (1-\lambda)(x-y) \in \Omega$ for all $\lambda \in [0,1]$. Therefore we can use the trick

$$\begin{aligned}f(x) - f(y) &= \left| \int_0^1 Df(y + \tilde{\lambda}(x-y)) \underbrace{(x-y)}_{\text{chain rule}} \,\mathrm{d}\tilde{\lambda} \right| \\ &\leqslant \left| \int_0^1 Df(y + \tilde{\lambda}(y-x) \,\mathrm{d}\tilde{\lambda} \right| |x-y| \\ &< L|x-y|\end{aligned}$$

and the claim follows. □

1.8 Prove the generalized Hölder inequality: if $\sum_{i=1}^n \frac{1}{p_i} = 1$ and $f_i \in L^{p_k}(\Omega)$ then the product $f_1 \dots f_n \in L^1(\Omega)$, with

$$\int_\Omega |f_1(x) \dots f_n(x)| \,\mathrm{d}x \leqslant \|f_1\|_{L^{p_1}} \dots \|f_n\|_{L^{p_n}}.$$

Proof. The base case $n = 2$ is already given by Hölder inequality. For the inductive step, assume the inequality holds for any k numbers. Now let $p_1, \dots, p_{k+1}$ be given with sum of reciprocal 1. If we define p to be such that $1/p = 1/p_k + 1/p_{k+1}$, then $1 = p/p_k + p/p_{k+1}$ and applying the normal Hölder inequality gives

$$\|f_k f_{k+1}\|_{L^p}^p = \|f_k^p f_{k+1}^p\|_{L^1} \overset{(H)}{\leqslant} \|f^p\|_{L^{p_k/p}} \|f^p\|_{L^{p_{k+1}/p}} = \|f_k\|_{L^{p_k}}^p \|f_{k+1}\|_{L^{p_{k+1}}}^p,$$

so $\|f_k f_{k+1}\|_{L^p} \leqslant \|f_k\|_{L^{p_k}} \|f_{k+1}\|_{L^{p_{k+1}}}$. Using our induction hypothesis with $\sum_{i=1}^{k-1} \frac{1}{p_i} + \frac{1}{p} = 1$,

$$\begin{aligned}\int_\Omega |f_1(x) \dots f_{k+1}(x)| \mathrm{d}x &\leqslant \|f_1\|_{L^1} \dots \|f_{k-1}\|_{L^{p_{k-1}}} \|f_k f_{k+1}\|_{L^p} \\ &\leqslant \|f_1\|_{L^1} \dots \|f_{k-1}\|_{L^{p_{k-1}}} \|f_k\|_{L^{p_k}} \|f_{k+1}\|_{L^{p_{k+1}}},\end{aligned}$$

which finishes the proof. □

1.9 Use Hölder's inequality to obtain the L^p interpolation inequality,

$$\|u\|_{L^p} \leqslant \|u\|_{L^q}^{q(r-p)/p(r-q)} \|u\|_{L^r}^{r(p-q)/q(r-q)},$$

where $q < p < r$ and $u \in L^r(\Omega)$.

Proof. Notice that $\frac{r-p}{r-q} + \frac{p-q}{r-q} = 1$. It follows that

$$\begin{aligned} p = r - (r-p) &= r - (r-q)\frac{r-p}{r-q} \\ &= q \cdot \frac{r-p}{r-q} + r - r \cdot \frac{r-p}{r-q} \\ &= q \cdot \frac{r-p}{r-q} + r \cdot \frac{p-q}{r-q}, \end{aligned}$$

Therefore

$$|u|^p = (|u|^q)^{(r-p)/(r-q)} (|u|^r)^{(p-q)/(r-q)}.$$

Then by Hölder inequality we have

$$\begin{aligned} \|u\|_{L^p}^p = \int_\Omega |u(\tilde{x})|^p \, \mathrm{d}\tilde{x} &\leqslant \left(\int_\Omega |u(\tilde{x})|^q \, \mathrm{d}\tilde{x} \right)^{(r-p)/(r-q)} \left(\int_\Omega |u(\tilde{x})|^r \, \mathrm{d}\tilde{x} \right)^{(p-q)/(r-q)} \\ &= \|u\|_{L^q}^{q(r-p)/(r-q)} \|u\|_{L^r}^{r(p-q)/(r-q)}. \end{aligned}$$

Taking p^{th} root of both sides gives us the L^p interpolation inequality. □

1.10 Show that given $s \in S(\Omega)$, $p \in [1, \infty)$, and $\epsilon > 0$, there exists an $f \in C_c^0(\Omega)$ such that

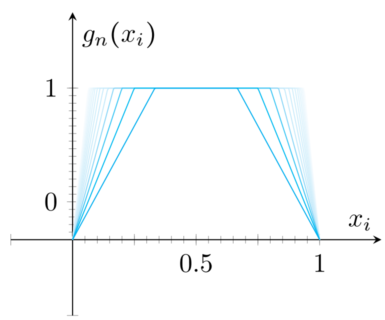
$$\|f - s\|_{L^p} < \epsilon.$$

Proof. Assume $\Omega \subset \mathbb{R}^m$. By definition $s \in S(\Omega)$ is of form

$$s(x) = \sum_{j=1}^n c_j \chi[I_j](x) \text{ where } I_j = \prod_{k=1}^m [a_i, b_i], \text{ i.e., boxes in } \mathbb{R}^m.$$

(i is the index of the component in $\mathbb{R}^m$.) or each $[a_i, b_i]$, consider the following approximation by g_n. The diagram on the right is an illustration with $[a, b] = [0, 1]$:

$$g_n(x_i) = \begin{cases} n(x_i - a_i), & x_i \in [a_i, a_i + 1/n], \\ 1, & x_i \in (a_i + 1/n, b_i - 1/n), \\ n(b_i - x_i) & x_i \in [b_i - 1/n, b_i]. \end{cases}$$



It follows that as $n \to \infty$, $g_n(x_i) \to \chi[a_i, b_i]$, and so

$$f_n := \prod_{i=1}^m c_i g_n(x_i) \to \prod_{i=1}^n c_i \chi[I_i](x) \text{ in } L^p(\Omega) \text{ if } p < \infty.$$

We just need to pick n large enough such that $\|f_n - s\|_{L^p} < \epsilon$. It's very easy to verify that $f_n \in C_c^0(\Omega)$: its compact support is simply $\prod_{i=1}^m [a_i, b_i]$. Then the claim follows. □

1.15 Show that $\ell^2(\Gamma)$ is not separable if Γ is uncountable, where $\ell^2(\Gamma)$ consists of real-valued functions on Γ with

$$\sum_{\gamma\in\Gamma} |f(\gamma)|^2 < \infty$$

and the inner product is defined by

$$\|f, g\|_{\ell^2(\Gamma)} := \sum_{\gamma\in\Gamma} f(\gamma)g(\gamma).$$

Proof. Assume Γ is uncountable. Consider the uncountable family of "sequences"

$$\{e^{(i)}\}_{i\in\Gamma} \subset \Gamma \text{ defined by } e^{(i)}_j := \delta_{ij}.$$

It follows that for all distinct $i, j \in \Gamma$, $\|e^{(i)} - e^{(j)}\|_{\ell^2(\Gamma)} = \sqrt{2}$. If Γ admits a countable dense subset, then for all $B(e^{(i)}, \sqrt{2}/2)$, some element from that dense subset must lie within. Then some element in this countable dense subset must be simultaneously in at least two of the open balls. But the open balls are disjoint! Therefore Γ cannot be separable. □