2.1 Show that the contraction mapping theorem remains true if the assumption that h is a contraction is replaced by that h^n is a contraction for $n > 1$.

Proof. If h^n is a contraction, then by contraction mapping theorem there exists a fixed point $x \in X$ satisfying $h^n(x) = x$. Applying h to both sides, we have

$$h(h^n(x)) = h(x) \implies h^n(h(x)) = h(x),$$

and so $h(x)$ is also a fixed point of h^n. By uniqueness $h(x) = x$, i.e., x is a fixed point for h. □

2.2 Show that one cannot replace the condition of contraction mapping theorem with

$$\|h(x) - h(y)\| < \|x - y\|$$

unless X is compact.

Proof. First, as an counterexample, we present a function $f : \mathbb{R} \to \mathbb{R}$ whose derivative < 1 everywhere. The function is constructed based on the *Signoid function* $s(x) = 1/(1 + e^{-x})$, a strictly increasing function $\mathbb{R} \to (0, 1)$. Taking the antiderivative of $1 - s(x)$ we have

$$f(x) = \int 1 - \frac{1}{1 + e^{-\tilde{x}}} \, \mathrm{d}\tilde{x} = x - \ln(x^x + 1) + C.$$

Setting $C = 0$ we indeed have a function satisfying $\|f(x) - f(y)\| < \|x - y\|$ but has no fixed point.

Now that a counterexample for noncompact X has been provided, it remains to show the claim still holds if X is compact. Suppose h does not admit a fixed point, then $\|h(x) - x\| > 0$ for all $x \in X$. Since h and id_X are both continuous, so is the mapping $g(x) : x \mapsto \|h(x) - x\|$. Since X is compact, g attains its minimum, say $\epsilon > 0$, so there exists x_0 satisfying $\|h(x_0) - x_0\| = \epsilon$. But then

$$\|h(h(x_0)) - h(x_0)\| < \|h(x_0) - x_0\| = \epsilon,$$

contradicting the minimality of $g(x_0)$. Hence h must have a fixed point. Uniqueness follows from the same argument given in the standard theorem: if x, y are both fixed points then

$$\|h(x) - h(y)\| = \|x - y\| < \|x - y\| \iff x = y.$$ □

2.3 If X is compact, use Ex.1.2 to show that one can find a countable set $\{x_i\}$ and an increasing sequence of integers $\{N_i\}$ such that

$$|x - x_i| \leqslant 2^{-n} \text{ for some } 1 \leqslant i \leqslant N_n.$$

Proof. By the existence of *finite ϵ-nets*, given 2^{-n} there exists a finite set $x_1^1, x_1^2, \dots, x_1^{M_1}$ that approximates any $x \in X$ with a distance $< 2^{-n}$. Taking the union of all such points (while letting n vary), we obtain a countable set

$$\{x_1^1, \dots, x_1^{M_1}, x_2^1, \dots, x_2^{M_2}, \dots\}$$

that satisfies the problem's requirement, with $N_n = M_1 + \dots + M_n$. □

2.4 Assuming for simplicity that f is globally bounded, use the Arzelá-Ascoli theorem to show that even if the solutions of $\mathrm{d}x/\mathrm{d}t = f(x)$ with $x(0) = x_0$ are not unique, the set of all possible $\{x(\tau)\}$,

$$X_\tau = \{y : \text{ there is a solution } x(t) \text{ with } x(\tau) = y\}$$

is closed.

Proof. Suppose we have a sequence $\{y_n\}$ with $x_n(\tau) = y_n$ that converge to some $\tilde{y}$ (x_n's are assumed to have compact domain $[0, \tau]$). We want to show that there exists some solution $\tilde{x}$ (of f) such that $\tilde{x}(\tau) = \tilde{y}$. Since f is globally bounded, we know

$$x_n(t) = x_n(0) + \int_0^t f(x_n(s))\,\mathrm{d}s \implies |x_n(t)| \leqslant \|x_0\| + t\|f\|_\infty \leqslant \|x_0\| + \tau\|f\|_\infty$$

$$\implies \sup_{t\in[0,\tau]} |x_n(t)| \text{ is uniformly bounded for all } n.$$

On the other hand, x_n's are also uniformly Lipschitz (and thus equicontinuous) with Lipschitz constant $\|f\|_\infty$. Therefore Arzelá-Ascoli applies and some subsequence $\{x_{n_i}\}$of $\{x_n\}$ converges uniformly to some x^*. Then, applying the uniform convergence $\{x_{n_i}\} \to x^*$ to

$$x_{n_i}(t) = x_0 + \int_0^t f(x_{n_i}(s))\,\mathrm{d}s,$$

we have

$$x^*(t) = x_0 + \int_0^t f(x(s))\,\mathrm{d}s,$$

so indeed x^* is a solution to the initial condition given by the problem and $x^*(\tau) = \tilde{y}$ as desired. □

2.5 Suppose that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x|^2 \leqslant C(t)|x|,$$

where $C(t)$ is continuous. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t_+}|x| \leqslant C(t).$$

Proof. The claim immediately follows from chain rule if $|x(t)| \neq 0$. Now assume that for some t_0 we have $|x(t_0)| = 0$. Notice that showing the desired inequality is equivalent to showing

$$\frac{\mathrm{d}}{\mathrm{d}t_+}|x(t_0)| \leqslant C(t_0) + \epsilon \text{ for all } \epsilon > 0.$$

Now let $\epsilon > 0$ be given. By the continuity of $C(t)$, there exists $\delta > 0$ such that $C(t) \leqslant C(t_0) + \epsilon$ for all $t \in [t_0, t_0 + \epsilon)$. Then,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|^2 \leqslant (C(t_0) + \epsilon)|x|.$$

Stuck at here. I tried to apply the differential inequality lemma (2.7) but didn't find any use of it. $\not\square$

2.6 Prove that if $a(t)$ is increasing and $x(t) \geqslant 0$ satisfies

$$x(t) \leqslant a(t) + \int_0^t b(\tilde{t})x(\tilde{t})\,\mathrm{d}\tilde{t}$$

then

$$x(t) \leqslant a(t)\exp\left(\int_0^t b(s)\,\mathrm{d}s\right).$$

Hint: consider the new variable $y(t) = \int_0^t b(s)x(s)\,\mathrm{d}s$ *and integrate the equation for* $\mathrm{d}y/\mathrm{d}t$.

Proof. Define $y(t)$ according to the hint. Then $x(t) \leqslant a(t) + y(t)$ and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = b(t)x(t) \leqslant b(t)\left[a(t) + y(t)\right].$$

Leaving the term containing $a(t)$ on the RHS, we obtain $\mathrm{d}y/\mathrm{d}t - b(t)y(t) \leqslant a(t)b(t)$. Since exp is always positive, we also have

$$\left[\frac{\mathrm{d}y}{\mathrm{d}t} - b(t)y(t)\right]\exp\left(-\int_0^t b(s)\,\mathrm{d}s\right) \leqslant a(t)b(t)\exp\left(-\int_0^t b(s)\,\mathrm{d}s\right). \tag{Δ}$$

Notice that the LHS is simply the derivative of $y(t)\exp(-\int b(s))$.

It suffices to show that, for all t^* defined for x,

$$x(t^*)a(t^*)\exp\left(\int_0^{t^*} b(s)\,\mathrm{d}s\right).$$

Pick an arbitrary t^* and fix it. For all $t \in [0, t^*]$, the monotonicity of $a(t)$ implies that the RHS of (Δ) is further bounded by $a(T)b(t^*)\exp(-\int_0^t b(s))$. Then, integrating both sides from 0 to t^* (with respect to t) gives

$$y(t^*)\exp\left(-\int_0^{t^*} b(s)\,\mathrm{d}s\right) \leqslant a(t^*)\int_0^{t^*}\left[b(\tilde{t})\exp\left(-\int_0^{\tilde{t}} b(s)\,\mathrm{d}s\right)\right]\mathrm{d}\tilde{t}.$$

Notice that

$$-\int_0^{\tilde{t}} b(s)\,\mathrm{d}s + \int_0^{t^*} b(s)\,\mathrm{d}s = \int_{\tilde{t}}^{t^*} b(s)\,\mathrm{d}s,$$

so

$$\begin{aligned} y(t^*) &\leqslant a(t^*)\int_0^{t^*}\left[b(\tilde{t})\exp\left(\int_{\tilde{t}}^{t^*} b(s)\,\mathrm{d}s\right)\right]\mathrm{d}\tilde{t} \\ &\leqslant a(t^*)\left[\exp\left(\int_0^{t^*} b(s)\,\mathrm{d}s\right) - \exp(0)\right] \end{aligned}$$

and the claim follows from the given inequality $x(t^*) \leqslant a(t^*) + y(t^*)$. □

2.7 Suppose that f is a globally Lipschitz function with constant L and that $g(x)$ is a continuous function with $\|f - g\|_\infty < \infty$. If $x(t)$ is the solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad x(0) = x_0$$

and $y(t)$ is any one of the solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(y), \qquad y(0) = x_0,$$

show, using lemmas 2.8 and 2.9, that

$$|x(t) - y(t)| \leqslant \frac{\|f - g\|_\infty}{Le^{Lt}}.$$

Proof. Define $z(t) := x(t) - y(t)$ and thus $\mathrm{d}z/\mathrm{d}t = f(x) - g(y)$. Lemma 2.9 then gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_+}|z| &\leqslant |f(x) - g(y)| \leqslant |f(x) - f(y)| + |f(y) - g(y)| \\ &\leqslant L|x - y| + \|f - g\|_\infty = L|z| + \|f - g\|_\infty. \end{aligned}$$

Since $x(0) = y(0)$, by the remark of lemma 2.9 (or directly applying Gronwall's inequality, 2.8), we obtain

$$|z(t)| \leqslant \left(|z(0)| + \frac{\|f - g\|_\infty}{L}\right)e^{Lt} \implies |x(t) - y(t)| \leqslant \frac{\|f - g\|_\infty}{L}e^{Lt}.$$ □