Problem 2

Show that the integration operator

$$I[f](x) = \int_0^x f(s)\,\mathrm{d}s, \qquad x \in [0,1]$$

is a bounded operator from $C^0([0,1])$ into itself. Show that it is also a bounded operator acting on $C^0([0,1])$ as a subset of $L^2(0,1)$ into $L^2(0,1)$.

Proof. For the first statement, the continuity of $I[f](x)$ is guaranteed by FTC. Furthermore, since $f \in C^0([0,1])$ it is bounded, say absolutely by M. Then

$$\|I[f]\|_\infty \leqslant \int_0^1 \|f\|_\infty\,\mathrm{d}x = \|f\|_\infty.$$

For the second argument (L^2), if $f \in L^2$ then by Cauchy-Schwarz

$$\begin{aligned}\|I[f]\|_2^2 = \int_0^1 |I[f](x)|^2\,\mathrm{d}x &= \int_0^1 \left(\int_0^x f(s)\,\mathrm{d}s\right)^2\,\mathrm{d}x \\ &\leqslant \int_0^1 \left(\int_0^x 1\,\mathrm{d}s\right)\left(\int_0^x |f(t)|^2\,\mathrm{d}t\right)\,\mathrm{d}x \\ &= x\int_0^1 \|f\|_2^2\,\mathrm{d}x \leqslant \|f\|_2^2.\end{aligned}$$

□

Problem 5

Suppose that $\{\varphi_j(x)\}$ is an orthonormal basis for $L^2(\Omega)$. Show that $\{\varphi_i(x)\varphi_j(y)\}$ is an orthonormal basis for $L^2(\Omega\times\Omega)$ and hence that, if $k \in L^2(\Omega\times\Omega)$, it can be written in the form

$$\|k\|_{L^2(\Omega\times\Omega)}^2 = \int_{\Omega\times\Omega} |k(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y = \sum_{i,j=1}^{\infty} |k_{i,j}|^2$$

where

$$k_{i,j} = \int_{\Omega\times\Omega} k(x,y)\varphi_i(x)\varphi_j(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. First notice that $\{\varphi_i(x)\varphi_j(y)\}$ indeed form an orthonormal subset of $L^2(\Omega\times\Omega)$:

$$\int_{\Omega\times\Omega} [\varphi_{i_1}(x)\varphi_{j_1}(y)][\varphi_{i_2}(x)\varphi_{j_2}(y)]\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \varphi_{i_1}(x)\varphi_{i_2}(x)\,\mathrm{d}x \int_\Omega \varphi_{j_1}(y)\varphi_{j_2}(y)\,\mathrm{d}y$$

which simply evaluates to $\delta_{i_1,i_2}\cdot\delta_{j_1,j_2}$ since $\{\varphi_i\}$ forms an orthonormal basis of Ω. Notice that if $k(x,y) \in L^2(\Omega\times\Omega)$ then $k(x,\cdot) \in L^2(\Omega)$ (i.e., first fix some y and treat k as a function of x). Therefore, fixing any $y \in \Omega$ and treating k as a function of x only, we can expand $k(x,y)$ by

$$k(x,y) = \sum_{i=1}^{\infty} u_i(y)\varphi_i(x) \text{ where } u_i(y) = \int_\Omega k(x,y)\varphi_i(x)\,\mathrm{d}x. \tag{1}$$

Our next goal is to expand $u_i(y)$ using $\{\varphi_j\}$. Indeed, this is well-defined because

$$\begin{aligned}\|u_i\|_2^2 = \int_\Omega |u_i(y)|^2 \, \mathrm{d}y &= \int_\Omega \left(\int_\Omega k(x,y)\varphi_i(x) \, \mathrm{d}x \right)^2 \, \mathrm{d}y \\ &\leqslant \int_\Omega \left(\int_\Omega k(x,y)^2 \, \mathrm{d}x \right) \left(\int_\Omega \varphi_i(x)^2 \, \mathrm{d}x \right) \, \mathrm{d}y \\ &= \int_{\Omega\times\Omega} |k(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y = \|k\|_2^2\end{aligned}$$

and thus $u_i(y) \in L^2(\Omega)$. Therefore, it also admits an expansion

$$\int_\Omega k(x,y)\varphi_i(x) \, \mathrm{d}x = u_i(y) = \sum_{j=1}^{\infty} \left(\int_\Omega u_i(y)\varphi_j(y) \, \mathrm{d}y \right) \varphi_j(y). \tag{2}$$

Substituting (2) into (1), we get

$$\begin{aligned}k(x,y) = \sum_{i=1}^{\infty} u_i(y)\varphi_i(x) &= \sum_{i=1}^{\infty} \left(\int_\Omega k(x,y)\varphi_i(x) \, \mathrm{d}x \right) \varphi_i(x) \\ &= \sum_{i=1}^{\infty} \left[\sum_{j=1}^{\infty} \left(\int_\Omega u_i(y)\varphi_j(y) \, \mathrm{d}y \right) \varphi_j(y) \right] \varphi_i(x) \\ &= \sum_{i=1}^{\infty} \left[\sum_{j=1}^{\infty} \int_\Omega \left(\int_\Omega k(x,y)\varphi_i(x) \, \mathrm{d}x \right) \varphi_j(y) \, \mathrm{d}y \, \varphi_j(y) \right] \varphi_i(x) \\ &= \sum_{i,j=1}^{\infty} \left(\int_{\Omega\times\Omega} k(x,y)\varphi_i(x)\varphi_j(y) \, \mathrm{d}x \, \mathrm{d}y \right) \varphi_i(x)\varphi_j(y),\end{aligned}$$

as desired. □

Problem 6

This is a partial converse of the Hilbert-Schmidt theorem. Show that if A can be expressed in the form

$$Au = \sum_{n=1}^{\infty} \lambda_n (u, w_n) w_n$$

where $\lambda_n \to 0$ and $(w_n, w_m) = \delta_{m,n}$ then A is compact and symmetric. [Hint: Theorem 3.10 & Lemma 3.12.]

Proof. We first define a sequence $\{A_n\}$ by the partial sums

$$A_n u = \sum_{i=1}^{n} \lambda_i (u, w_i) w_i.$$

The range of A_n has dimension n so each A_n is compact by Lemma 3.12. It remains to show that $A_n \to A$ in $\|\cdot\|_{\mathrm{op}}$, after which the compactness of A follows from Theorem 3.10. Indeed,

$$\|Au - A_n u\| = \left\| \sum_{i=n+1}^{\infty} \lambda_i (u, w_i) w_i \right\| \leqslant \left(\sup_{i \geqslant n+1} \lambda_i \right) \left\| \sum_{i=n+1}^{\infty} (u, w_i) w_i \right\| \leqslant \left(\sup_{i=n+1} \lambda_i \right) \|u\| \to 0$$

as $\lambda_n \to 0$ and $\sup\limits_{i \geqslant n+1} \lambda_i \to 0$. To see A is symmetric, notice that

$$\begin{aligned}(u, Av) = \left(u, \sum_{i=1}^{\infty} \lambda_i (v, w_i) w_i \right) &= \sum_{i=1}^{\infty} \lambda_i \Big(u, (v, w_i) w_i \Big) = \sum_{i=1}^{\infty} \lambda_i (u, w_i)(v, w_i) \\ &= \sum_{i=1}^{\infty} \lambda_i (u, w_i)(w_i, v) = \sum_{i=1}^{\infty} \lambda_i \Big((u, w_i) w_i, v \Big) = (Au, v).\end{aligned}$$

□