Problem 3.2

Show that the integration operator

$$I[f](x) = \int_0^x f(s)\,\mathrm{d}s, \qquad x \in [0,1]$$

is a bounded operator from $C^0([0,1])$ into itself. Show that it is also a bounded operator acting on $C^0([0,1])$ as a subset of $L^2(0,1)$ into $L^2(0,1)$.

Proof. For the first statement, the continuity of $I[f](x)$ is guaranteed by FTC. Furthermore, since $f \in C^0([0,1])$ it is bounded, say absolutely by M. Then

$$\|I[f]\|_\infty \leqslant \int_0^1 \|f\|_\infty\,\mathrm{d}x = \|f\|_\infty.$$

For the second argument (L^2), if $f \in L^2$ then by Cauchy-Schwarz

$$\begin{aligned}\|I[f]\|_2^2 = \int_0^1 |I[f](x)|^2\,\mathrm{d}x &= \int_0^1 \left(\int_0^x f(s)\,\mathrm{d}s\right)^2 \mathrm{d}x \\ &\leqslant \int_0^1 \left(\int_0^x 1\,\mathrm{d}s\right)\left(\int_0^x |f(t)|^2\,\mathrm{d}t\right) \mathrm{d}x \\ &= x\int_0^1 \|f\|_2^2\,\mathrm{d}x \leqslant \|f\|_2^2.\end{aligned}$$

□

Problem 3.5

Suppose that $\{\varphi_j(x)\}$ is an orthonormal basis for $L^2(\Omega)$. Show that $\{\varphi_i(x)\varphi_j(y)\}$ is an orthonormal basis for $L^2(\Omega\times\Omega)$ and hence that, if $k \in L^2(\Omega\times\Omega)$, it can be written in the form

$$\|k\|^2_{L^2(\Omega\times\Omega)} = \int_{\Omega\times\Omega} |k(x,y)|^2\,\mathrm{d}x\,\mathrm{d}y = \sum_{i,j=1}^{\infty} |k_{i,j}|^2$$

where

$$k_{i,j} = \int_{\Omega\times\Omega} k(x,y)\varphi_i(x)\varphi_j(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Proof. First notice that $\{\varphi_i(x)\varphi_j(y)\}$ indeed form an orthonormal subset of $L^2(\Omega\times\Omega)$:

$$\int_{\Omega\times\Omega} [\varphi_{i_1}(x)\varphi_{j_1}(y)][\varphi_{i_2}(x)\varphi_{j_2}(y)]\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \varphi_{i_1}(x)\varphi_{i_2}(x)\,\mathrm{d}x \int_\Omega \varphi_{j_1}(y)\varphi_{j_2}(y)\,\mathrm{d}y$$

which simply evaluates to $\delta_{i_1,i_2}\cdot\delta_{j_1,j_2}$ since $\{\varphi_i\}$ forms an orthonormal basis of Ω. Notice that if $k(x,y) \in L^2(\Omega\times\Omega)$ then $k(x,\cdot) \in L^2(\Omega)$ (i.e., first fix some y and treat k as a function of x). Therefore, fixing any $y \in \Omega$ and treating k as a function of x only, we can expand $k(x,y)$ by

$$k(x,y) = \sum_{i=1}^{\infty} u_i(y)\varphi_i(x) \text{ where } u_i(y) = \int_\Omega k(x,y)\varphi_i(x)\,\mathrm{d}x. \tag{1}$$

Our next goal is to expand $u_i(y)$ using $\{\varphi_j\}$. Indeed, this is well-defined because

$$\begin{aligned}\|u_i\|_2^2 = \int_\Omega |u_i(y)|^2 \, \mathrm{d}y &= \int_\Omega \left(\int_\Omega k(x,y)\varphi_i(x) \, \mathrm{d}x \right)^2 \, \mathrm{d}y \\ &\leqslant \int_\Omega \left(\int_\Omega k(x,y)^2 \, \mathrm{d}x \right) \left(\int_\Omega \varphi_i(x)^2 \, \mathrm{d}x \right) \mathrm{d}y \\ &= \int_{\Omega\times\Omega} |k(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y = \|k\|_2^2\end{aligned}$$

and thus $u_i(y) \in L^2(\Omega)$. Therefore, it also admits an expansion

$$\int_\Omega k(x,y)\varphi_i(x) \, \mathrm{d}x = u_i(y) = \sum_{j=1}^\infty \left(\int_\Omega u_i(y)\varphi_j(y) \, \mathrm{d}y \right) \varphi_j(y). \tag{2}$$

Substituting (2) into (1), we get

$$\begin{aligned}k(x,y) = \sum_{i=1}^\infty u_i(y)\varphi_i(x) &= \sum_{i=1}^\infty \left(\int_\Omega k(x,y)\varphi_i(x) \, \mathrm{d}x \right) \varphi_i(x) \\ &= \sum_{i=1}^\infty \left[\sum_{j=1}^\infty \left(\int_\Omega u_i(y)\varphi_j(y) \, \mathrm{d}y \right) \varphi_j(y) \right] \varphi_i(x) \\ &= \sum_{i=1}^\infty \left[\sum_{j=1}^\infty \int_\Omega \left(\int_\Omega k(x,y)\varphi_i(x) \, \mathrm{d}x \right) \varphi_j(y) \, \mathrm{d}y \, \varphi_j(y) \right] \varphi_i(x) \\ &= \sum_{i,j=1}^\infty \left(\int_{\Omega\times\Omega} k(x,y)\varphi_i(x)\varphi_j(y) \, \mathrm{d}x \, \mathrm{d}y \right) \varphi_i(x)\varphi_j(y),\end{aligned}$$

as desired. □

Problem 3.6

This is a partial converse of the Hilbert-Schmidt theorem. Show that if A can be expressed in the form

$$Au = \sum_{n=1}^\infty \lambda_n (u, w_n) w_n$$

where $\lambda_n \to 0$ and $(w_n, w_m) = \delta_{m,n}$ then A is compact and symmetric. [Hint: Theorem 3.10 *&* Lemma 3.12.]

Proof. We first define a sequence $\{A_n\}$ by the partial sums

$$A_n u = \sum_{i=1}^n \lambda_i (u, w_i) w_i.$$

The range of A_n has dimension n so each A_n is compact by Lemma 3.12. It remains to show that $A_n \to A$ in $\|\cdot\|_{\text{op}}$, after which the compactness of A follows from Theorem 3.10. Indeed,

$$\|Au - A_n u\| = \left\| \sum_{i=n+1}^\infty \lambda_i (u, w_i) w_i \right\| \leqslant \left(\sup_{i\geqslant n+1} \lambda_i \right) \left\| \sum_{i=n+1}^\infty (u, w_i) w_i \right\| \leqslant \left(\sup_{i=n+1} \lambda_i \right) \|u\| \to 0$$

as $\lambda_n \to 0$ and $\sup_{i\geqslant n+1} \lambda_i \to 0$. To see A is symmetric, notice that

$$\begin{aligned}(u, Av) = \left(u, \sum_{i=1}^\infty \lambda_i (v, w_i) w_i \right) &= \sum_{i=1}^\infty \lambda_i \Big(u, (v, w_i) w_i \Big) = \sum_{i=1}^\infty \lambda_i (u, w_i)(v, w_i) \\ &= \sum_{i=1}^\infty \lambda_i (u, w_i)(w_i, v) = \sum_{i=1}^\infty \lambda_i \Big((u, w_i) w_i, v \Big) = (Au, v).\end{aligned}$$

□

Problem 3.8

If $k \in L^2(\Omega \times \Omega)$ and $k(x,y) = k(y,x)$, show that the solution of the integral equation

$$\int_\Omega k(x,y)u(y)\,\mathrm{d}y = f(x), \qquad f \in L^2(\Omega)$$

is given in terms of the eigenvalues and eigenfunctions of the equation

$$\int_\Omega k(x,y)u(y)\,\mathrm{d}y = \lambda u(x)$$

by

$$u(x) = \sum_{i=1}^{\infty} \frac{(f,u_i)}{\lambda_i} u_i(x).$$

[Hint: consider the integral operator K defined by $[Ku](x) = \int_\Omega k(x,y)u(y)\,\mathrm{d}y$.]

Proof. Proposition 3.13 proved that $K : L^2(\Omega) \to L^2(\Omega)$ is compact and, with the assumption that $k(x,y) = k(y,x)$, Lemma 3.16 says K is symmetric. Therefore we may invoke the Hilbert-Schmidt Theorem: the eigenvalues $\{\lambda_i\}$ of K are real and $\lambda_i \to 0$ after relabeling. We define $u_i(x)$ to be the corresponding eigenfunction to λ_i. By definition,

$$[Ku_i](x) = \int_\Omega k(x,y)u_i(y)\,\mathrm{d}y = \lambda_i u_i(x).$$

The Hilbert-Schmidt Theorem also states that if $f \in L^2(\Omega)$ then $f = \sum_{i=1}^{\infty}(f,u_i)u_i$. Therefore, the solution to

$$\int_\Omega k(x,y)u(y)\,\mathrm{d}y = [Ku](x) = f(x) = \sum_{i=1}^{\infty}(f,u_i)u_i(x)$$

is simply given by $u(x) = \sum_{i=1}^{\infty} \frac{(f,u_i)}{\lambda_i} u_i(x)$: indeed,

$$[Ku](x) = \sum_{i=1}^{\infty} \frac{[Ku_i](x)(f,u_i)}{\lambda_i} = \sum_{i=1}^{\infty}(f,u_i)u_i(x) = f(x).$$ □

Problem 3.9

Show that if A is a linear operator that is bounded below, i.e., there exists $k > 0$ such that $\|Ax\| \geqslant k\|x\|$, then A^{-1} is well-defined. Show also that if A is bounded then $A^{-1} : R(A) \to D(A)$ is bounded.

Proof. If A is bounded below the $\ker(A) = \{0\}$ for if $x \neq y$ then $\|Ax - Ay\| \geqslant k\|x - y\| > 0$. If A is bounded then

$$\|y\|_Y = \|A(A^{-1}y)\|_Y \geqslant k\|A^{-1}y\|_X \implies \|A^{-1}y\| \leqslant \frac{\|y\|}{k} \implies \|A^{-1}\| \leqslant \frac{1}{k}.$$ □

Problem 3.10

Show that the sectorial operator defined by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At}\,\mathrm{d}t$$

agrees with the fraction powers of A presented in Section 3.10

$$A^{\alpha}u = \sum_{i=1}^{\infty} \lambda_i^{\alpha}(u, w_i)w_i$$

when H has a basis of eigenfunctions of A.
[Hint: apply $A^{-\alpha}$ to each eigenfunction and use the definition $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,\mathrm{d}t$.]

Proof. Let $\{w_i\}$ be the set of eigenfunctions with w_i corresponding the eigenvalue e_i. The matrix exponential gives $e^{A}w = e^{\lambda}w$ and $e^{-At}w = e^{-\lambda t}w$. Therefore, for any eigenfunction w_i,

$$A^{-\alpha}w_i = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-At}\,\mathrm{d}t\; w_i = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-\lambda_i t}\,\mathrm{d}t\; w_i.$$

Notice that u-substitution with $u := \lambda_i t$ gives

$$\int_0^{\infty} t^{\alpha-1}e^{-\lambda_i t}\,\mathrm{d}t = \int_0^{\infty} (u/\lambda_i)^{\alpha-1}e^{-u}\lambda_i^{-1}\,\mathrm{d}u = \int_0^{\infty} u^{\alpha-1}\lambda_i^{-\alpha}e^{-u}\,\mathrm{d}u = \lambda_i^{-\alpha}\Gamma(\alpha).$$

Therefore

$$A^{-\alpha}w_i = \frac{1}{\Gamma(\alpha)}\lambda_i^{-\alpha}\Gamma(\alpha)w_i = \lambda_i^{-\alpha}w_i,$$

i.e., $\lambda_i^{-\alpha}$ is an eigenvalue of $A^{-\alpha}$. Since w_i is chosen randomly, we are able to recover all eigenvalues of $A^{-\alpha}$ in this way. Corollary 3.26 then gives a representation of A, which is exactly of form presented in Section 3.10:

$$A^{-\alpha}u = \sum_{i=0}^{\infty} \lambda_i^{-\alpha}(u, w_i)w_i \implies A^{\alpha}u = \sum_{i=1}^{\infty} \lambda_i^{\alpha}(u, w_i)w_i.$$ □

Problem 3.11

Prove that if u is in the domain of A^k then

$$\|A^s u\| \leqslant \|A^{\ell}u\|^{(k-s)/(k-l)}\|A^k u\|^{(s-\ell)/(k-\ell)}$$

where $0 \leqslant \ell < s < k$. While proving, use the eigenfunction expansion $u = \sum_{i=1}^{\infty} c_i w_i$ and the corresponding expression for $\|A^s u\|$:

$$\|A^s u\|^2 = \sum_{i=1}^{\infty} \lambda_i^{2s}|c_i|^2.$$

Also make use of Hölder's inequality.

Proof. This one really reminds me of the L^p interpolation inequality (Ex.1.9). Notice that the two exponents on the RHS add up to 1. I will try to work backwards in the proof so that each step looks more reasonable than just some random magical trick. The main idea is to show that the square of the original LHS, i.e., $\|A^s u\|^2$, does not exceed that of the RHS. Notice again that $(k-s)/(k-\ell) + (s-\ell)/(k-\ell) = 1$: this cries out for Hölder's inequality for ℓ^p spaces. Here we need $p := (k-\ell)/(k-s)$ and $q := (k-\ell)/(s-\ell)$. Then, for a, b under certain conditions (which we will specify later),

$$\begin{aligned}
\text{LHS}^2 = \|A^s u\|^2 &= \sum_{i=1}^{\infty} \lambda_i^{2s} |c_i|^2 = \sum_{i=1}^{\infty} \left(\lambda_i^{2a} |c_i|^{2/p}\right)\left(\lambda_i^{2b}|c_i|^{2/q}\right) \\
&\leqslant \sum_{i=1}^{\infty} \left| \left(\lambda_i^{2a} |c_i|^{2/p}\right)\left(\lambda_i^{2b}|c_i|^{2/q}\right) \right| \\
&\leqslant \left(\sum_{i=1}^{\infty} \lambda_i^{2ap} |c_i|^{2p/p}\right)^{1/p} \left(\sum_{j=1}^{\infty} \lambda_j^{2bq} |c_J|^{2q/q}\right)^{1/q} \\
&= \left(\sum_{i=1}^{\infty} \lambda_i^{2ap} |c_i|^{2}\right)^{1/p} \left(\sum_{j=1}^{\infty} \lambda_i^{2bq} |c_j|^{2}\right)^{1/q} \\
&= \|A^{ap} u\|^{2/p} \|A^{bp} u\|^{2/q} \\
\text{(want to have)} &\stackrel{?}{=} \|A^{\ell} u\|^{2/p} \, \|A^{k} u\|^{2/q} = \text{RHS}^2.
\end{aligned}$$

From the "backward thinking" above, we see that there are three conditions that a and b need to satisfy:

$$a + b = s \qquad ap = \frac{a(k-\ell)}{k-s} = \ell \qquad \text{and} \qquad bq = \frac{b(k-\ell)}{s-\ell} = k$$

A simple substitution suggests that

$$a = \frac{\ell(k-s)}{k-\ell} \text{ and } b = \frac{k(s-\ell)}{k-\ell}$$

is a solution. Therefore the proof is valid and

$$\|A^s u\|^2 \leqslant \|A^\ell u\|^{2(k-s)/(k-\ell)} \|A^k u\|^{2(s-\ell)/(k-\ell)} \implies \|A^s u\| \leqslant \|A^\ell u\|^{(k-s)/(k-\ell)} \|A^k u\|^{(s-\ell)/(k-\ell)}.$$ □

Problem 3.12

Recall that if A is a positive symmetric operator with compact inverse, we have

$$e^{-At} u = \sum_{i=1}^{\infty} e^{-\lambda_i t} (u, w_i) w_i.$$

Using this equality, show that if $x \in D(A)$ then $e^{-At}x$ is differentiable on $[0, \infty)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{-At} x = -A e^{-At} x.$$

[Hint: it suffices to check differentiability at $t = 0$.]

Proof. Let $x \in D(A)$ be given and let $x := \sum_{i=1}^{\infty} u_i w_i$. We want to show $\frac{\mathrm{d}}{\mathrm{d}t} e^{-At} x = -A e^{-At} x$ and, by the hint, it suffices to check $t = 0$. Thus it suffices to show

$$\lim_{h \downarrow 0} \frac{(e^{-Ah} - I)x}{h} = -AIx.$$

By the Hilbert-Schmidt Theorem,

$$\begin{aligned}\frac{(e^{-Ah}-I)x}{h}+Ax &= \sum_{i=1}^{\infty}\frac{(e^{-\lambda_i h}-1)u_i w_i}{h}+\sum_{i=1}^{\infty}\lambda_i u_i w_i \\ &= \sum_{i=1}^{\infty}\left[\frac{e^{-\lambda_i h}-1}{h}+\lambda_i\right]u_i w_i \\ &= \sum_{i=1}^{\infty}\left[\frac{e^{-\lambda_i h}-1}{\lambda_i h}+1\right]\lambda_i u_i w_i.\end{aligned}$$

Since A is positive, so are all its eigenvalues λ_i; also, h is positive. Therefore the cyan term is always strictly between 0 and 1:

$$\lim_{t\downarrow 0}\frac{e^{-t}-1}{t}=-1,\ \lim_{t\to\infty}\frac{e^{-t}-1}{t}=0,\ \text{and}\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{e^{-t}-1}{t}\right]=\frac{e^{-t}(e^t-t-1)}{t^2}>0 \text{ for all } t>0$$

so $(e^{-\lambda_i h}-1)/(\lambda_i h)\in(-1,0)$ and the cyan term $\in(0,1)$. Therefore,

$$\left\|\sum_{i=n}^{\infty}\left[\frac{e^{-\lambda_i h}-1}{\lambda_i h}+1\right]\lambda_i u_i w_i\right\| \leqslant \sum_{i=n}^{\infty}\lambda_i^2 u_i^2 \to 0 \qquad (\Delta)$$

because $x\in D(A)$ and Corollary 3.26 (in particular equation (3.20)) gives $\sum_{i=1}^{\infty}\lambda_i^2|u_i|^2<\infty$.

Let $\epsilon>0$ be given. It follows that there exists $N\in\mathbb{N}$ such that the LHS of (Δ) is less than $\epsilon/2$ whenever $n\geqslant N$. Now fix any such n. Since $\lim_{t\downarrow 0}(e^{-t}-1)/t=-1$, for sufficiently small h we have

$$\left\|\sum_{i=1}^{n-1}\left[\frac{e^{-\lambda_i h}-1}{\lambda_i h}+1\right]\lambda_i u_i w_i\right\| < \frac{\epsilon}{2}.$$

Therefore

$$\lim_{h\downarrow 0}\frac{(e^{-Ah}-I)x}{h}=Ax \implies \frac{\mathrm{d}}{\mathrm{d}t}e^{-At}x\bigg|_{t=0}=-Ae^{-At}x\bigg|_{t=0}. \qquad \square$$