Robinson, Chapter 4 Exercises

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Problem 1

[MATH 580 HW1 revisited; now I feel much better than three months ago] Use Zorn's lemma to show that every Hilbert space has a basis. *Hint: find a maximal orthonormal set.*

Proof. Let H be a Hilbert space and let P denote the set of all orthonormal subsets of H. We define a partial ordering on P by inclusion. If $\{P_i\}$ is a chain where $i \in I$ (some index set), i.e., $P_n \subset P_m$ for $n < m$, then the union of all P_i's in this chain, i.e., $S := \bigcup_{i \in I} P_i$, is an upper bound for $\{P_i\}$. It follows that S is also orthonormal. That all elements in S have norm 1 is clear. For orthogonality, suppose $u, v \in S$ are not orthogonal. Then since u, v are both contained in some P_n we get a contradiction on the orthogonality of P_n.

Now we claim that S is a basis for H. If not, $H \setminus \text{span}(S)$ is nonempty. In particular, since $\text{span}(S)$ is a closed linear subspace of H, for $x \in H \setminus \text{span}(S)$ we are able to orthogonally decompose it into $x_1 + x_2$ where $x_1 \in \text{span}(S)$ and $x_2 \in (\text{span}(S))^\perp$. Hence $(\text{span}(S))^\perp$ is nonempty and we have found a way to extend the supposedly maximal orthonormal subset S of H, contradiction. Thus S is indeed a basis for H. □

Problem 2

Show if Y is a proper linear subspace of X then there exists a nonzero element of X^* that vanishes on Y.

Proof. Clearly we want to invoke the Hahn-Banach theorem, so it suffices to find a larger subspace Z (compared to Y) and a functional $f \in X^*$ such that f vanishes on Y but not all of Z. Such construction is easy: we begin by taking $z \in X \setminus Y$ and consider the linear subspace $Z := \{y + \lambda z \mid y \in Y, \lambda \in \mathbb{K}\}$. Notice that any $w \in Z$ can be written <u>uniquely</u> as $w = y + kz$ because

$$w = y_1 + k_1 z = y_2 + k_2 z \implies y_2 - y_1 = (k_1 - k_2) z,$$

where the LHS is in Y and the RHS is in $\text{span}\{z\}$, and the only possibility is if $y_1 = y_2$ and $k_1 = k_2$. Therefore the following functional f on Z is well defined:

$$f(w) = f(y + kz) = k.$$

It is also easy to verify that f vanishes on Y but does not vanish entirely on Z. Then $\tilde{f}$ obtained by extending f to all of X via Hahn-Banach is an nonzero element of X^*, proving the claim. □

Problem 3

For Ω of finite volume, show if $f \in L^\infty(\Omega)$ then the norm of the linear functional L_f defined on $L^1(\Omega)$ by

$$L_f(g) = \int_\Omega f(x)g(x)\,\mathrm{d}x \qquad \text{for all} \qquad g \in L^1(\Omega)$$

satisfies

$$\|L_f\|_{(L^1)^*} = \|f\|_{L^\infty}.$$

Hint: consider the sequence of functions $g_p(x) = |f(x)|^{p-2}f(x)$ and use Proposition 1.16.

Proof. [This is highly analogous to the example presented in section 4.2.1 except we replaced $p, q \in (1, \infty)$ by $p = 1, q = \infty$.] The direction $\|L_f\|_{(L^1)^*} \leqslant \|f\|_{L^\infty}$ is immediate by Hölder's inequality:

$$|L_f(g)| \leqslant \|f\|_\infty \|g\|_1 \implies \|L_f\|_* \leqslant \|f\|_\infty.$$

It remains to show the other direction. Since there is no such thing as $|f(x)|^{\infty-2}f(x)$, we instead consider

$$g_p(x) = |f(x)|^{p-2}f(x).$$

Similar to the example in text,

$$\|g_p\|_1 = \int_\Omega |f(x)|^{p-1}\,\mathrm{d}x = \|f\|_{p-1}^{p-1}$$

and

$$|L_f(g_p)| = \left|\int_\Omega |f(x)|^{p-2}f(x)^2\,\mathrm{d}x\right| = \left|\int_\Omega |f(x)|^p\,\mathrm{d}x\right| = \|f\|_p^p.$$

Therefore $\|L_f\|_* \geqslant \|f\|_p^p / \|f\|_{p-1}^{p-1}$. Letting $p \to \infty$ we have

$$\|L_f\|_* \geqslant \lim_{p\to\infty} \frac{\|f\|_p^p}{\|f\|_{p-1}^{p-1}} = \frac{\lim_{p\to\infty}\|f\|_p^p}{\lim_{p\to\infty}\|f\|_{p-1}^{p-1}} = \lim_{p\to\infty}\frac{\|f\|_\infty^p}{\|f\|_\infty^{p-1}} = \|f\|_\infty.$$ □

Problem 4

Use the Rises representation theorem to prove the Hahn-Banach theorem for a Hilbert space.

Proof. Let M be a linear subspace of a Hilbert space$H$. As M inherits the inner product on H, it is also Hilbert. Therefore given a functional f on M there exists (a unique) $m \in M$ such that

$$f(x) = (m, x) \text{ for all } x \in M.$$

Notice that Cauchy-Schwarz gives $|f(x)| \leqslant \|m\|\|x\| \implies \|f\|_* \leqslant \|m\|$, and letting f act on m itself gives $|f(m)| = \|m\|^2$ and so $\|f\|_* \geqslant \|m\|$. Therefore $\|f\|_* = \|m\|$. *This is Lemma 7.7 in MATH 580.*
Now if we simply define $\tilde{f}$, a functional on all of H, by

$$\tilde{f}(h) = (m, h) \text{ for all } h \in H,$$

it becomes clear that $\tilde{f}$ extends f and $\|\tilde{f}\|_{H^*} = \|f\|_{M^*}$ (same reasoning as above). □

Problem 5

Suppose that M is a linear subspace of a Banach space X and that $\{x_n\}$ is a sequence of elements of M that converges weakly in X to some x. Show that $x \in M$. Deduce that

$$x = \sum_{i=1}^{\infty} c_i x_i$$

for some coefficients $\{c_i\}$. *Hint: as a first step show that if $f(x) = 0$ for every $f \in X^*$ such that $f\big|_M = 0$, then $x \in M$.*

Proof. If $x \notin M$ then we can invoke the result of problem 2 and consider the linear span of M and x. This (along with Hahn-Banach) would give is $f \in X^*$ that vanishes on M but not at x. But since $x_n \rightharpoonup x$, for this particular f we must have

$$0 = \lim_{n\to\infty} f(x_n) = f(x) \neq 0$$

which is absurd. Hence $x \in M$.

In addition to $x \in M$, we can show analogously that $x \in \operatorname{span}\{x_n\}$, and this proves the second claim on $\{c_i\}$. □

Problem 6

If $x_n \in C^0([a,b])$ and $x_n \rightharpoonup x$ in $C^0([a,b])$, show that $\{x_n\}$ is pointwise convergent on $[a,b]$.

Proof. In particular, for $t \in [a,b]$, consider $f_t \in (C^0([a,b]))^*$ defined by $f_t(x) = x(t)$. Since $[a,b]$ is compact and x continuous, we immediately see f_t is bounded. Linearity is trivial. Hence f_t is indeed in $(C^0([a,b]))^*$. By weak convergence $x_n \rightharpoonup x$ we see $f_t(x_n) \to f_t(x)$, i.e., $x_n(t) \to x(t)$. Since t is chosen arbitrarily, $x_n \to x$ pointwise ever were on $[a,b]$. □

Problem 7

Let H be Hilbert. Show that if $x_n \rightharpoonup x$ in H and $\|x_n\| \to \|x\|$ then $x_n \to x$.

Proof. First we rewrite $\|x_n - x\|^2$ as $\|x_n\|^2 + \|x\|^2 - (x_n, x) - (x, x_n)$. Since inner product is a continuous mapping, that $x_n \rightharpoonup x$ implies $(x_n, x) \to (x,x)$ and $(x, x_n) \to (x,x)$. Of course, by assumption $\|x_n\|^2 \to \|x\|^2$ as well. Thus this entire expression $\to \|x\|^2 + \|x\|^2 - (x,x) - (x,x) = 0$, i.e., $x_n \to x$ strongly. □