Chapter 5 (Partial) Exercises

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Problem 1

Show that if $u \in \mathcal{D}'(\Omega)$ and $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ then

$$\langle D^\alpha u, \varphi_n \rangle \to \langle D^\alpha u, \varphi \rangle .$$

Proof. By definition $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ implies $D^\alpha \varphi_n \to D^\alpha \varphi$, and

$$\langle D^\alpha u, \varphi_n \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi_n \rangle \to (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle = \langle D^\alpha u, \varphi \rangle .$$

(The equalities are given on page 113.) □

Problem 2

For $\psi \in C_c^\infty(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, we can define the distribution ψu by

$$\langle \psi u, \varphi \rangle := \langle u, \psi\varphi \rangle \text{ for all } \varphi \in C_c^\infty(\Omega).$$

Show that we do indeed have $\psi u \in \mathcal{D}'(\Omega)$ and that

$$D(u\psi) = uD\psi + \psi Du.$$

Proof. Let $\{\varphi_n\} \to \varphi$ in $\mathcal{D}(\Omega)$. By (5.9) $\langle u, \varphi_n \rangle \to \langle u, \varphi \rangle$, so

$$\langle \psi u, \varphi_n \rangle = \langle u, \psi\varphi_n \rangle \to \langle u, \psi\varphi \rangle = \langle \psi u, \varphi \rangle ,$$

and

$$\begin{aligned} \langle D(u\psi), \varphi \rangle &= -\langle u\psi, D\varphi \rangle = -\langle u, \psi D\varphi \rangle \\ &= -\langle u, \psi D\varphi + \varphi D\psi - \varphi D\psi \rangle \\ &= -\langle u, D(\psi\varphi) \rangle + \langle u, \varphi D\psi \rangle = \langle Du, \psi\varphi \rangle + \langle u, \varphi D\psi \rangle \\ &= \langle \psi Du + uD\psi, \varphi \rangle . \end{aligned}$$ □

Problem 4

Prove that, under the conditions of Propitiation 5.8, there exists a constant $C(k)$ such that

$$\|u\|_{H^k}^2 \leqslant C \sum_{|\alpha|=k} |D^\alpha u|^2 \quad \text{for all} \quad u \in H_0^k(\Omega).$$

(Hint: induction.)

Proof. Poincaré's inequality (Proposition 5.8) already gives the base case $k = 1$. Now assume

$$\|u\|_{H^n}^2 \leqslant C \sum_{|\alpha|=n} |D^\alpha u|^2$$

for all $u \in H_0^n(\Omega)$. By definition $\|u\|_{H^{n+1}}^2 = \|u\|_{H^n}^2 + \sum_{|\alpha|=n+1} |D^\alpha u|^2$, so it suffices to show that

$$\|u\|_{H^{n+1}}^2 \leqslant C_n \sum_{|\alpha|=k} |D^\alpha u|^2 + \sum_{|\alpha|=n+1} |D^\alpha u|^2$$

for all $u \in H_0^{n+1}(\Omega)$ [where C_n is the coefficient in our induction hypothesis for case $k = n$]. By lemma 5.10,

$$u \in H_0^{n+1}(\Omega) \implies D^\alpha u \in H_0^1(\Omega)$$

so Poincaré's inequality applies, giving us

$$|D^\alpha u| \leqslant C|D(D^\alpha u)| \quad \text{for all} \quad D^\alpha u \in H_0^1(\Omega).$$

Note that any multi-index β with $|\beta| = n + 1$ satisfies $D^\beta u = D(D^\alpha u)$ for some $|\alpha| = 1$, and we are done. □

Problem 5

Show that if $\psi \in C_c^\infty(\Omega)$ and $u \in H^k(\Omega)$ then $\psi u \in H^k(\Omega)$ and

$$\|\psi u\|_{H^k(\Omega)} \leqslant C(\psi)\|u\|_{H^k(\Omega)}.$$

Proof. Let $\{u_n\} \subset C^\infty(\Omega)$ be a sequence whose limit is u. Since

$$\begin{aligned} |D^\alpha(\psi u_n)| &= \left| \sum_{|\beta| \leqslant |\alpha|} \binom{\alpha}{\beta} D^\beta \psi D^{\alpha-\beta} u_n \right| \\ &\leqslant \sum_{|\beta| \leqslant |\alpha|} \binom{\alpha}{\beta} |D^\beta \psi| |D^{\alpha-\beta} u_n| \\ &= \underbrace{\left(\sum_{|\beta| \leqslant |\alpha|} \binom{\alpha}{\beta} |D^\beta \psi| \right)}_{\text{finite}} \|u_n\|_{H^k(\Omega)}, \end{aligned}$$

all the derivatives of $\{\psi u_n\}$ up to order k are convergent, and so is their sum. Therefore $\psi u \in H^k(\Omega)$ with

$$\|\psi u\|_{H^k}(\Omega) \leqslant \left(\sum_{|\beta| \leqslant |\alpha|} \binom{\alpha}{\beta} |D^\beta \psi| \right) \|u\|_{H^k(\Omega)}.$$ □

Problem 6

Show that the unbounded function
$$f := \log\log\left(1+\frac{1}{|x|}\right)$$
is still an element of $H^1(B(0,1))$ where $B(0,1)$ is the unit ball of $\mathbb{R}^2$.

Proof. We first show that f is indeed in $L^2(B(0,1))$:
$$\int_{B(0,1)}\left[\log\log\left(1+\frac{1}{|x|}\right)\right]^2\,\mathrm{d}x = \int_0^{2\pi}\int_0^1\left[\log\log\left(1+\frac{1}{r}\right)\right]^2 r\,\mathrm{d}r\mathrm{d}\theta.$$
This integral is clearly finite when $r \nrightarrow 0$, so it suffices to check its behavior near 0:
$$r\left[\log\log\left(1+\frac{1}{r}\right)\right]^2 \sim 2\left[\log\log\left(1+\frac{1}{r}\right)\right]\cdot\frac{1}{\log(1+1/r)}\frac{1}{1+1/r}\frac{1}{-r^2}(-r^2)^{-1} \to 0$$
by L'Hôpital's rule. Thus it is L^2. The partial derivative(s) is(are) given by
$$\frac{\partial f}{\partial x_i} = -\frac{1}{\log(1+1/|x|)}\frac{x_i}{|x|^2(1+|x_i|)}$$
(through writing them as x, y in $\mathbb{R}^2$ might have been better). Therefore
$$\begin{aligned}\int_{B(0,1)}|Df(x)|^2\,\mathrm{d}x &= \int_0^{2\pi}\int_0^1\frac{1}{r(1+r)^2\log(1+1/r)^2}\,\mathrm{d}r\mathrm{d}\theta\\ &= 2\pi\int_0^{1/2}\frac{1}{[\dots]}\,\mathrm{d}r + 2\pi\int_{1/2}^1\frac{1}{[\dots]}\,\mathrm{d}r\\ &\leqslant 2\pi\int_0^{1/2}\frac{1}{r(\log r)^2}\,\mathrm{d}r + 2\pi\int_{1/2}^1\frac{1}{[\dots]}\,\mathrm{d}r\\ &\leqslant -\frac{2\pi}{\log u}\bigg|_{u=0}^{1/2} + 2\pi\int_{1/2}^1\frac{1}{r(1+r)^2(\log r)^2}\,\mathrm{d}r.\end{aligned}$$
Clearly the second integral is finite, so the entire integral is finite, i.e., $f \in H^1(B(0,1))$. □

Problem 7

Shoe that if $\Omega \subset \mathbb{R}^3$ is a bounded C^1 domain then for $u \in H^1(\Omega)$
$$\|u\|_{L^3} \leqslant C|u|^{1/2}\|u\|_H^{1/2}$$
where $|u| = |u|_2 = \|u\|_{L^2(\Omega)}$. Hint: (5.33).

Proof. Using (5.33), since $[u(x)]^2 = 2\int_{-\infty}^{x_i} uD_iu\,\mathrm{d}y_i$ for $i = 1,2,3$, we have
$$|u(x)|^2 \leqslant 2\int_{-\infty}^{\infty}|uD_iu|\,\mathrm{d}y_i$$
and so
$$|u(x)|^3 \leqslant \left(8\int_{-\infty}^{\infty}|uD_1u|\,\mathrm{d}y_1\int_{-\infty}^{\infty}|uD_2u|\,\mathrm{d}y_2\int_{-\infty}^{\infty}|uD_3u|\,\mathrm{d}y_3\right)^{1/2}. \tag{Δ}$$
The result will follow from integrating (Δ) iteratively. We will adopt the book's notation in which the constant

C may vary from line to line, and we will highlight the variables it depends on, if any.

$$\int |u(x)|^3 \, \mathrm{d}x_1 \leqslant C \left(\int |uD_1u| \, \mathrm{d}y_1 \right)^{1/2} \left(\iint |uD_2u| \, \mathrm{d}x_1 \mathrm{d}y_2 \right)^{1/2} \left(\iint |uD_3u| \, \mathrm{d}x_1 \mathrm{d}y_3 \right)^{1/2},$$

$$\int |u(x)|^3 \, \mathrm{d}x_1 \mathrm{d}x_2 \leqslant C \left(\iint |uD_1u| \, \mathrm{d}y_1 \mathrm{d}x_2 \right)^{1/2} \left(\iint |uD_2u| \, \mathrm{d}x_1 \mathrm{d}y_2 \right)^{1/2} \left(\iint |uD_3u| \, \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_3 \right)^{1/2},$$

and finally the RHS evolves to a product related to three triple integrals, namely

$$\int_\Omega |u(x)|^3 \, \mathrm{d}x \leqslant C \prod_{i=1}^{3} \left(\int_\Omega |uD_iu| \, \mathrm{d}x \right)^{1/2}.$$

Since $\int_\Omega |uD_iu| \, \mathrm{d}x \leqslant |u||D_iu|$ we obtain

$$\|u\|_{L^3}^3 \leqslant C|u|^{3/2} \cdot \prod_{i=1}^{3} |D_iu|^{1/2} \leqslant C|u|^{3/2}|Du|^{3/2}.$$

Just like in Lemma 5.27, we first use the density of $C_c^1(\Omega') \subset H_0^1(\Omega)$ to claim $\|u\|_{L^3} \leqslant C|u|^{1/2}|Du|^{1/2}$ for $u \in H_0^1(\Omega')$ and then use the extension theorem to prove it for the entire $H^1(\Omega)$. □

Problem 8

(Corollary 5.30) Show that if $\Omega \Subset \mathbb{R}^m$ is a bounded C^k domain and $k > (m/2) + j$ then each $u \in H^k(\Omega)$ is an element of $C^j(\overline{\Omega})$ with

$$\|u\|_{C^j} \leqslant C\|u\|_{H^k}.$$

Proof. We simply apply Theorem 5.29 to all $D^\alpha u$ where $|\alpha| \leqslant j$. Then it follows that $D^\alpha u \in C^0(\overline{\Omega})$ and

$$\|D^\alpha u\|_\infty \leqslant C(m,k,j)\|u\|_{H^{k-j}} \leqslant C(m,k,j)\|u\|_{H^k}.$$

In particular, some α shows that $u \in C^j(\overline{\Omega})$ and $\|u\|_C \leqslant C\|u\|_{H^k}$. □

Problem 9

Let V be the subspace of $H^1(\Omega)$ consisting of functions with zero integral over Ω:

$$V = \{u \in H^1(\Omega) : \int_\Omega u(x) \, \mathrm{d}x = 0\}.$$

Argue by contradiction, using Kellich-Kondrachov Compactness Theorem (Theorem 5.32), that there exists a constant C that gives the Poincaré inequality:

$$|u| \leqslant C|\nabla u| \qquad \text{for all} \qquad u \in V.$$

Hint: you may assume that if $Du = 0$ then u is constant a.e.

Proof. Suppose for contradiction that there exists a sequence $\{u_n\} \subset V$ with $|u_n| \geqslant n|\nabla u_n|$. Normalizing each u_n gives a sequence $\{v_n\}$ such that $|\nabla v_n| \leqslant 1/n$. Invoking theorem 5.32 (since $\{v_n\}$ is bounded in $H^1(\Omega)$), $\{v_n\}$ must converge to some function $v \in V$ with norm 1 and integral 0 over Ω. Per the hint, since for any test function

$\varphi \in \mathcal{D}(\Omega)$

$$\langle v, D\varphi \rangle = \lim_{n\to\infty} \langle v_n, D\varphi \rangle = -\lim_{n\to\infty} \langle Dv_n, \varphi \rangle = 0,$$

we have $Dv = 0$, i.e., v is constant a.e. But if this were true, then it's impossible that

$$|v| = 1 \qquad \text{and} \qquad \int_\Omega v(x)\,\mathrm{d}x = 0$$

simultaneously. We have obtained a contradiction so the Poincaré inequality must hold. □

Problem 11

Prove that if $u \in H_p^1(Q)$ then $|u| \leqslant (L/2\pi)|Du|$.

Proof. Writing u in terms of its Fourier coefficient, $u = \sum_{k=\mathbb{Z}^m} c_k e^{2\pi i k x/L}$, we obtain

$$Du = \sum_{k\in\mathbb{Z}^m} \frac{2\pi i k}{L} c_k e^{2\pi i k x/L}.$$

By Parseval's identity,

$$|u|^2 = L^m \sum_{k\in\mathbb{Z}^m} |c_k|^2 \qquad \text{and} \qquad |Du|^2 = L^m \sum_{k\in\mathbb{Z}^m} \left[\frac{4\pi^2}{L^2}|c_k^2||k^2|\right],$$

and indeed we have $|u| \leqslant (L/2\pi)|Du|$. □