MATH 501 Problem Set 10

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5.2.4 Prove that if A is Hermitian, then the deflation technique in the text will produce a Hermitian matrix.

Proof. Since A is Hermitian, $A = A^*$. Therefore

$$(UAU^*)^* = U^{**}A^*U^* = UA^*U^* = UAU^*,$$

i.e., the "bigger matrix" containing the deflation matrix $\hat{A}$ is Hermitian. Since $\hat{A}$ and A share the same diagonal, it is therefore also Hermitian. □

5.2.6 A **normal** matrix is one that commutes with it conjugate transpose: $AA^* = A^*A$. Prove that if A is normal then so is $A - \lambda I$ for any scalar λ.

Proof. Notice that we have $(A - \lambda I)^* = A^* - \overline{\lambda} I$. Thus,

$$\begin{aligned}(A-\lambda I)^*(A-\lambda I) &= (A^* - \overline{\lambda}I)(A-\lambda I) = A^*A - \lambda A^* - \overline{\lambda}A + |\lambda|^2 I \\ &= (A-\lambda I)(A^* - \overline{\lambda}I) = (A-\lambda I)(A-\lambda I)^*.\end{aligned}$$

□

5.2.7 Suppose that A is normal and that x and y are eigenvectors of A corresponding to different eigenvalues. Prove that $x^*y = 0$.

Proof. Suppose $Ax = \lambda_1 x$ for $x \neq 0$. From above we know that $(A - \lambda_1 I)$ is normal, and thus

$$\|(A-\lambda_1 I)x\| = x^*(A-\lambda_1 I)^*(A-\lambda_1 I)x = x^*(A-\lambda_1 I)(A-\lambda_1 I)^*x = \|(A-\lambda_1 I)^*x\|$$

for all x. This means that $(A-\lambda_1 I)^*x = (A^* - \overline{\lambda}_1 I)x = 0$ and thus x is also an eigenvalue of A^* with eigenvector $\overline{\lambda}_1$. Now suppose also $Ay = \lambda_2 y$ with $\lambda_1 \neq \lambda_2$. We have $A^*y = \overline{\lambda}_2 y$ and

$$\begin{aligned}(\lambda_1 - \lambda_2)(x^*y) &= \lambda_1 x^*y - x^*\lambda_2 y = (\overline{\lambda}_1 x)^*y - x^*(\lambda_2 y) \\ &= (A^*x)^*y - x^*(Ay) = x^*Ay - x^*Ay = 0.\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$, we conclude that $x^*y = 0$. □

5.2.8 Prove that if A then A and A^* have the same eigenvectors.

Proof. This has been shown in 5.2.7. □

5.2.10 Prove that if x and y are points in $\mathbb{C}^n$ having the same Euclidean norm, then there is a unitary matrix U such that $Ux = y$.

5.2.15 Prove that if A is a diagonal matrix, then $\|A\|_2 = \max\limits_{1\leqslant i\leqslant n}|a_{i,i}|$.

Proof. Suppose $a_{k,k}$ is the largest diagonal entry. Then

$$\begin{aligned}\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2 &= \sup_{\|x\|_2=1} \left(\sum_{i=1}^{n} a_{i,i}^2 x_i^2\right)^{1/2} \\ &\leqslant \sup_{\|x\|_2=1} \left(\sum_{i=1}^{n} a_{k,k}^2 x_i^2\right)^{1/2} \\ &\leqslant \sup_{\|x\|_2=1} |a_{k,k}| \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \\ &= |a_{k,k}| \sup_{\|x\|_2=1} \|x\|_2 = |a_{k,k}|.\end{aligned}$$

□

5.2.16 Prove that for any square matrix A, $\|A\|_2^2 \leqslant \|A^* A\|_2$.

Proof. Notice that by SVD, for x with $\|x\|_2 = 1$,

$$\sup_{\|x\|_2=1} \|Ax\|_2 = \sup_{\|x\|_2=1} \|U\Sigma V^T x\|_2 = \sup_{\|x\|_2=1} \|\Sigma V^T x\|_2 = \sup_{\|y\|_2=1} \|\Sigma y\|_2$$

since U and V^T are unitary and so $\|v\| = \|Uv\| = \|V^T v\|$. It follows that $\|\Sigma y\|_2$ can be maximized when $y = e^{(1)}$, and when this happens the norm evaluates to the largest singular value σ_1 of A. Thus $\|A\|_2^2 = \sigma_1^2$, and it is clear that $\|A^* A\|_2$ is also σ_1^2. Therefore they are equal and the inequality provided in the problem is trivially true. □

5.2.20 Prove that if the eigenvalues of A satisfy $|\lambda_1| > |\lambda_i|$ for $i = 2, 3, \ldots, n$, then

$$\lambda_1 = \lim_{m\to\infty} \operatorname{tr}(A^{m+1})/\operatorname{tr}(A^m).$$

Proof. Assuming that we can take it for granted that the trace of A is the sum of eigenvalues, the expression is equivalent to

$$\lambda_1 = \lim_{m\to\infty} \frac{\lambda_1^{m+1} + \cdots + \lambda_n^{m+1}}{\lambda_1^m + \cdots + \lambda_n^m}, \text{ or } \lim_{m\to\infty} \frac{(\lambda_1 - \lambda_2)\lambda_2^m + \cdots + (\lambda_1 - \lambda_n)\lambda_n^m}{\lambda_1^m + \cdots + \lambda_n^m} = 0,$$

since the eigenvalues of A^m are $\lambda_1^m, \ldots, \lambda_n^m$. Since λ_1 is strictly larger than all other λ_i's, as $m \to \infty$, the ratio $\lambda_i^m / \lambda_1^m$ tends to 0, and the claim follows. □

5.2.31 Find the Schur factorization $UAU^* = T$ for the matrix

$$A = \begin{bmatrix} 2.888 & 0.984 & -1.440 \\ 1.184 & 3.312 & -1.920 \\ -0.160 & 2.120 & -0.200 \end{bmatrix}$$

Solution

One eigenvalue of this matrix is $\lambda_1 = 1$ with unit eigenvector $x := [0.36 \quad 0.48 \quad 0.8]^T$. Consider

$$\alpha = \frac{\sqrt{2}}{\|x - e^{(1)}\|} = \frac{5}{4} \text{ and } v = \alpha(x - e^{(1)}) = [-0.8 \quad 0.6 \quad 1]^T.$$

Define $U := I - vv^* = \begin{bmatrix} 0.36 & 0.48 & 0.8 \\ 0.48 & 0.64 & -0.6 \\ 0.8 & -0.6 & 0 \end{bmatrix}$, and we have obtained Schur's factorization on this matrix:

$$UAU^* \approx \begin{bmatrix} 1 & 4 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

5.2.38 Use Gershgorin's Theorem to prove that a diagonally dominant matrix does not have zero as an eigenvalue and is therefore nonsingular.

Proof. The theorem says that if λ is an eigenvalue then it is in some D_i centered at $a_{i,i}$ with radius $\sum_{j \neq i} |a_{i,j}|$. By diagonal dominance, $|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$, so D_i cannot contain the origin. Notice that all D_i's have this property, and thus $\lambda \neq 0$. Hence A is nonsingular. □