MATH 501 Problem Set 12

Qilin Ye

April 25, 2021

Pseudocode Programming

Unlike what they have done in other chapters, Cheney and Kincaid did not include a sample pseudocode for SVD in Chapter 5.4. Therefore it is reasonable to assume that problem 5.4.40 allows the use of `pinv(A)`. If so, the solution is simply given by `x = pinv(A) * B`.

If, `pinv(A)` is not allowed in this assignment, the code on the right provides an alternate version using MATLAB's `svd(A)` only (so that the pseudoinverse is computed separately via $(U\Sigma V^T)^+ = V\Sigma^+ U^*$).

```
Command Window
>> [A_pseudo, x]

ans =

         0         0         0         0         0
   -0.4000    0.3000         0         0    0.1000
    0.6000    0.8000         0         0    8.6000
```

Left 3×4 is A^+, rightmost column is x

```
A = [0,-1.6,0.6;0,1.2,0.8; 0,0,0; 0,0,0];
b = [5; 7; 3; -2];

[U, Sigma, T] = svd(A);
[m, n] = size(Sigma);

Sigma_pseudo = zeros(n,m);
for i = 1: min(m,n)
    if Sigma(i,i) ~= 0
        Sigma_pseudo(i,i) = 1 / Sigma(i,i);
    end
end

A_pseudo = T * Sigma_pseudo * U';
```

Textbook Problems

5.4.3 Find A^+ in the case that AA^* is invertible.

Solution. Claim: if this is the case then $A^+ = A^*(AA^*)^{-1}$. To prove the claim, it suffices to check that $B = A^*(AA^*)^{-1}$ satisfies all four Penrose properties:

$$ABA = AA^*(AA^*)^{-1}A = A \tag{1}$$

$$BAB = A^*(AA^*)^{-1}AA^*(AA^*)^{-1} = A^*(AA^*)^{-1} = B \tag{2}$$

$$I = (AB)^* = (AA^*(AA^*)^{-1})^* = I^* \tag{3}$$

$$(BA)^* = (A^*(AA^*)^{-1}A)^* = A^*((AA^*)^{-1})^*A = A^*(AA^*)^{-1}A = BA \tag{4}$$

(where the second-last step in (4) uses the fact that AA^* is Hermitian.)

5.4.4 Find A^+ in the case that $A^*A = I$.

Solution. Claim: if $A^*A = I$ then $A^+ = A^*$. Indeed,

$$AA^*A = A \tag{1}$$
$$A^*AA^* = A^* \tag{2}$$
$$(AA^*)^* = A^{**}A^* = AA^* \tag{3}$$
$$(A^*A)^* = A^*A^{**} = A^*A. \tag{4}$$

5.4.5 Find A^+ in the case that A is Hermitian and idempotent, i.e., $A^* = A = A^2$.

Solution. Claim: the pseudoinverse is A itself. Indeed, if so we have

$$AA^+A = A^3 = A(A^2) = A^2 = A = A^3 = A^+AA^+ \qquad \text{(1) and (2)}$$

and

$$(AA^+)^* = (A^2)^* = A^* = AA^+ = A^* = (A^2)^* = (A^+A)^* \qquad \text{(3) and (4)}$$

5.4.6 Prove that if A is hermitian then so is A^+.

Proof. Just like $(A^T)^{-1} = (A^{-1})^T$ for invertible A, we now show that $(A^*)^+ = (A^+)^*$. Indeed, using the Penrose properties of A^+ on A, we obtain the following:

$$A^*(A^+)^*A^* = (AA^+A)^* = A^* \tag{1}$$
$$(A^+)^*A^*(A^+)^* = (A^+AA^+)^* = (A^+)^* \tag{2}$$
$$(A^*(A^+)^*)^* = A^+A = (A^+A)^* = A^*(A^+)^* \tag{3}$$
$$((A^+)^*A^*)^* = AA^+ = (AA^+)^* = (A^+)^*A^* \tag{4}$$

Therefore $(A^*)^+ = (A^+)^*$. Since A is Hermitian, $(A^*)^+ = A^+$ so $A^+ = (A^+)^*$, i.e., A^+ is Hermitian. □

5.4.9 If A is Hermitian, what is the relationship between its eigenvalues and its singular values?

Solution. If λ is an eigenvalue of A then it also is one for A^* (which is just A). Therefore λ^2 is an eigenvalue of $A^*A = A^2$ since

$$Ax = \lambda x \implies A^2x = A(Ax) = \lambda Ax = \lambda^2 x.$$

It follows that $|\lambda|$ is a singular value of A.

5.4.18 Prove that if A is Hermitian and positive definite then its eigenvalues are identical with its singular values.

Proof. If A is positive definite then all its eigenvalues must be positive, for any negative eigenvalue λ with eigenvector x would give

$$x^*Ax = x^*\lambda x = \lambda x^*x = \lambda\|x\|^2 < 0,$$

contradicting its positive definiteness. Then by the previous problem, λ_i would coincide with σ_i. □

5.4.20 Let A be an $n \times n$ matrix having singular values $\sigma_1, ..., \sigma_n$. Prove that the determinant of A is

$$\det(A) = \pm\sigma_1\sigma_2...\sigma_n.$$

Proof. Notice that $\det(A) = \det(A^*) = \sqrt{\det(A^*A)}$. Since the eigenvalues of A^*A are $\sigma_1^2, ..., \sigma_n^2$, it suffices to show that

$$\det(A^*A) = \prod_{i=1}^{n} \sigma_i^2.$$

More generally, we can show that the determinant of a matrix $M_{n\times n}$ is the product of all its eigenvalues. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of M. It follows that the characteristic polynomial of M is

$$\det(M - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda),$$

and setting $\lambda := 0$ proves our claim. □

5.4.21 Let $\|A\|_2$ denote the matrix norm subordinate to the Euclidean vector norm. Let σ_1 be the largest singular value of A. Show that $\|A\|_2 = \sigma_1$.

Proof. I've shown this in Ex.5.2.16 in HW10. By SVD,

$$\|A\|_2 = \sup_{\|x\|_2=1} \|Ax\|_2 = \sup_{\|x\|_2=1} \|U\Sigma V^* x\|_2 = \sup_{\|x\|_2=1} \|\Sigma V^* x\|_2 = \sup_{\|y\|_2=1} \|\Sigma y\|_2$$

where the third equality uses the fact that U is unitary, thus preserving the norm of $\Sigma V^* x$ and the last equality uses the fact that V^* is unitary and $\sup_{\|x\|_2=1}$ is the same as $\sup_{\|V^*x\|_2=1}$. It then becomes clear that $\|\Sigma y\|_2$ can be maximized when $y = e^{(1)}$, and when this happens, the norm evaluates to σ_1. Thus $\|A\|_2^2 = \sigma_1^2$ and $\|A\|_2 = \sigma_1$. □

5.4.30 Prove that if the $m \times n$ matrix A has rank n then $A^+ = (A^*A)^{-1}A^*$.

Proof. If A is $m \times n$ with rank n then A^*A is a symmetric $n \times n$ matrix with full rank:

$$A^*Ax = 0 \implies 0 = x^*A^*Ax = \|Ax\|^2 \implies x = 0.$$

Therefore the inverse $(A^*A)^{-1}$ is well-defined. Now it remains to verify the proposed A^+ satisfies all four Penrose properties:

$$AA^+A = A(A^*A)^{-1}A^*A = A \tag{1}$$

$$A^+AA^+ = (A^*A)^{-1}A^*A(A^*A)^{-1}A^* = (A^*A)^{-1}A^* = A^+ \tag{2}$$

$$(AA^+)^* = (A(A^*A)^{-1}A^*)^* = A^{**}((A^*A)^{-1})^*A^* = A(A^*A)^{-1}A^* = AA^+ \tag{3}$$

$$I^* = (A^+A)^* = ((A^*A)^{-1}A^*A)^* = I^* \tag{4}$$

This proves the claim. □