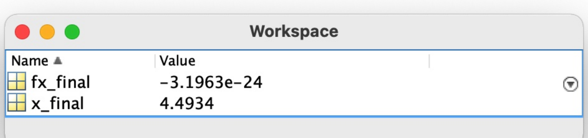
MATH 501 Problem Set 3

Pseudocode Programming

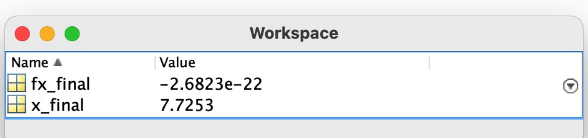
For solving a single-valued function (p.71 prob. 1):

```
syms f(x);
f(x) = x - tan(x);
g = diff (f);

x_old = input('Enter x0 here: ');

if (abs(f(x_old))<eps)
   x_final = double(x_old);
   fx_final = double(f(x_old));
   clearvars -except x_final fx_final
   return
end
for k = 1:100
   x_new = x_old - f(x_old) / g(x_old);
   v = f(x_new);
   if ((abs(x_new-x_old)<eps) || (abs(v)<eps))
      disp (k + 'iterations operated.');
      break
   else
      x_old = x_new;
   end
end

x_final = double(x_new);
fx_final = double(f(x_new));
clearvars -except x_final fx_final
```

Outputs for $x_0 = 4.5$ (4 iterations):



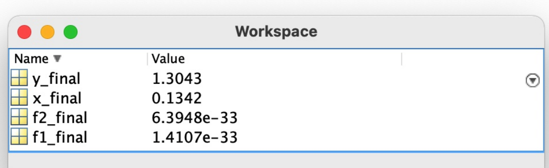
Outputs for $x_0 = 7.7$ (5 iterations):



For solving system of nonlinear equations (p.74 prob. 34):

```
syms x y
f1(x,y) = 4*y^2 + 4*y + 52*x - 19;
f2(x,y) = 169*x^2 + 3*y^2 + 111*x - 10*y - 10;

x_old = input('Enter initial x0: ');
y_old = input('Enter initial y0: ');

if (max(abs(f1(x_old,y_old)),abs(f2(x_old,y_old)))<eps)
    f1_value = f1(x_old,y_old);
    f2_value = f2(x_old,y_old);
    disp('Initial guess is good enough');
    clearvars
else
    for k = 1:100
        F = [x_old; y_old];
        Fx = [f1(x_old,y_old); f2(x_old,y_old)];
        J = jacobian([f1,f2],[x,y]);
        F_new = F - inv(J(x_old,y_old))*Fx;
        x_new = F_new(1,1);
        y_new = F_new(2,1);
        f1_value = f1(x_new,y_new);
        f2_value = f2(x_new,y_new);
        if(max([abs(f1_value),abs(f2_value),abs(x_new-x_old),
        abs(y_new-y_old)])<eps)
            disp (k + ' iterations operated.');
            break
        else
            x_old = x_new;
            y_old = y_new;
        end
    end
end
```

One solution pair with $x_0 = y_0 = 1$ and 7 iterations: (some extra code were included to convert values from `sym` back to `double` and to cleanup the workspace)



Textbook Problems

3.2.6 Consider **Steffensen's method** with the iteration formula

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}.$$

Show that this is quadratically convergent under suitable hypothesis.

Proof. To begin with, we assume f is "nice" in the sense that $f \in C^2$ and $f'(r) = f(r) = 0$. We show that $e_n := x_n - r$ converges to 0 with $e_{n+1} \sim e_n^2$.

By Taylor's approximation theorem, since $f \in C^2$ (and therefore in C^1), we have

$$f(x_n + f(x_n)) = f(x_n) + f'(x_n)f(x_n) + \frac{f''(\xi_1)}{2} f(x_n)^2 \tag{1}$$

$$f(x_n - e_n) = 0 = f(x_n) - e_n f'(x_n) + e_n^2 \frac{f''(\xi_2)}{2} \tag{2}$$

$$f(x_n - e_n) = 0 = f(x_n) - e_n f'(\xi_3) \tag{3}$$

for some ξ_1 between x_n and $x_n + f(x_n)$, some ξ_2, ξ_3 between r and x_n (the first two lines used $f \in C^2$ and the third used $f \in C^1$).

By (1), we can rewrite the Steffensen's method as

$$x_{n+1} = x_n - \frac{f(x_n)^1}{f'(x_n) + f''(\xi_1)f(x_n)/2},$$

and so since $e_{n+1} = x_{n+1} - r$,

$$\begin{aligned} e_{n+1} &= \overbrace{x_n - r}^{e_n} - \frac{f(x_n)}{f'(x_n) + f''(\xi_1)f(x_n)/2} \\ &= \frac{-f(x_n) + e_n f'(x_n) + e_n f''(\xi_1) f(x_n)/2}{f'(x_n) + f''(\xi_1)f(x_n)/2} \\ &= \frac{e_n^2 f''(\xi_2)/2 + e_n^2 f''(\xi_1) f'(\xi_3)/2}{f'(x_n) + f''(\xi_1)f(x_n)/2} \\ &= e_n^2 \cdot \left[\frac{f''(\xi_2) + f''(\xi_1)f'(\xi_3)}{2f'(x_n) + f''(\xi_1)f(x_n)}\right]. \end{aligned}$$

In fact, the monstrous coefficient coefficients cancels out nicely to $\left|\frac{f''(r)}{f'(r)}\right| \cdot \frac{|1 + f'(r)|}{2}$ as $x_n \to r$ and the continuity of x implies $\xi_1, \xi_2, \xi_3 \to r$. Indeed Steffensen's method is quadratically convergent. □

3.2.7 What is the purpose of the following iteration formula? Identify it as the Newton iteration for a certain function.

$$x_{n+1} = 2x_n - x_n^2 y$$

Solution

(This question is ambiguous; I here treat y as a constant, but since y is usually interpreted as $y = f(x)$, there of course exists a drastically different result if we think that way.) First rewrite this as $x_{n+1} = x_n - (x_n^2 y - x_n)$. Then $f(x)/f'(x) = x^2 y - x \implies f'(x) = f(x)/(x^2 y - x)$. Using $1/(x^2 y - x)$ as the integrating factor, one sees that

$$f(x) = \exp \int 1/(x^2 y - x)\,\mathrm{d}x = \exp[\ln((1-xy)/x) + C] = \frac{1-xy}{Cx}.$$

3.2.13 Devise a Newton iteration formula for computing $\sqrt[3]{R}$ where $R > 0$. Perform a graphical analysis of your function $f(x)$ to determine the starting values for which the iteration will converge.

Solution

We simply need to find the root of $f(x) = x^3 - R$. This function has derivative $f'(x) = 3x^2$. Then the iteration formula is given by

$$x_{n+1} =: g(x_n) = x_n - \frac{f(x)}{f'(x)} = x_n - \frac{x_n^3 - R}{3x_n^2} = \frac{2x_n}{3} + \frac{R}{3x_n^2}.$$

In fact, we first show that there are only countable many x_0's that would eventually iterate to 0. First notice that the only real solution to $x_{n+1} = 0$ is

$$-\frac{2x_n}{3} = \frac{R}{3x_n^2} \implies x_n^3 = -2R \implies x_n = -\sqrt[3]{R/2}.$$

Further notice that if $g(x) = -\sqrt[3]{R/2}$ then immediately $x < 0$. However, as $x < 0$, $g(x)$ is monotone. This means there will be a unique negative solution to $g(x) = -\sqrt[3]{R/2}$. So on and so forth. In the end, we obtain a list of real numbers which eventually iterate to $-\sqrt[3]{R/2}$ and thus 0. Other than these countably many x_0's, we will never iterate to 0, so Newton's method is well-defined except for these special cases.

If $\sqrt[3]{R} = r > 0$ and $x_0 > r$, then the sequence $\{x_n\}$ will always be positive since $2x_n/3$ and $R/3x_n^2$ are. On the other hand, since $(x_n^3 - R)/3x_n^2$ is positive, $\{x_n\}$ is also monotone decreasing. Therefore it converges to some $\tilde{r}$. Since the mapping $x_n \mapsto x_n - f(x)/f'(x)$ is continuous, the limit $\tilde{r}$ must be a fixed point, i.e.,

$$\tilde{r}^3 = \tilde{r}^3 - \frac{f(\tilde{r})}{f'(\tilde{r})} \implies f(\tilde{r}) = 0 \implies \tilde{r} = \sqrt[3]{R}.$$

Now suppose $r > 0$ and $x_0 < r$. It clearly follows that $\{x_n\}$ is monotone increasing. Should it be bounded above by something $< r$ it cannot converge to r, a contradiction. On the other hand, notice that $-r/2$ is the "magic point" where

$$g(-r/2) = -\frac{r}{2} - \frac{f(-r/2)}{f'(r/2)} = -\frac{r}{3} + \frac{r^3}{3r^2/4} = r.$$

If $x > -r/2$, it's easy to verify that $g(x) > r$ as the tangent line passing through $(x, x^3 - r^3)$ is less steep than the secant line between $(x, x^3 - r^3)$ and $(r, 0)$: the former has slope $3x^2$, whereas the second has slope $(r^3 - x^3)/(r - x) = r^2 + rx + x^2 > x^2 + x^2 + x^2$.

Therefore, as $\{x_n\}$ monotonously increases, some x_n has to reach or pass $-r/2$, resulting in $x_{n+1} > r$, and the claim follows from the first part, i.e., $r > 0$ and $x_0 > r$, treating $x_0 := x_{n+1}$.

To sum up, we've shown that most nonzero x_0's (except for a countably many exceptions) will eventually produce a sequence $\{x_n\}$ that converges to $\sqrt[3]{R}$.

3.2.15 The function $f(x) = x^2 + 1$ has zeros in the complex plane at $x = \pm i$. Is there a real starting point for the complex Newton's method such that the iteration converge to either of these zeros? Answer the same question for complex starting points.

Solution

No. If we start with a real number then $\{x_n\}$ is a subset of the real axis and we'll never get any complex x_n's out of it. I have little clue how to prove the convergence for the case of complex numbers, but after modifying my MATLAB program, every single complex x_0 I've tested converged to either i or $-i$, so I suspect the claim holds as well.

3.2.22 Which of the following converges quadratically?

$$(a)\frac{1}{n^2} \qquad (b)\frac{1}{2^{2^n}} \qquad (c)\frac{1}{\sqrt{n}} \qquad (d)\frac{1}{e^n} \qquad (e)\frac{1}{n^n}$$

Solution

Only (b) converges quadratically with $(1/2^{2^n})^2 = 1/2^{2^{n+1}}$. It's obvious that the rest don't:

(a) $\lim_{n\to\infty}(1/(n+1)^2)/(1/n^2)^2 = \lim_{n\to\infty} n^4/(n+1)^2 = \infty$.

(c) $\lim_{n\to\infty}(1/\sqrt{n+1})/(1/\sqrt{n})^2 = \lim_{n\to\infty} n/\sqrt{n+1} = \infty$.

(d) $\lim_{n\to\infty}(1/e^{n+1})/(1/e^n)^2 = \lim_{n\to\infty} e^{n-1} = \infty$.

(e) $\lim_{n\to\infty}(1/(n+1)^{n+1})/(1/n^n)^2 = \lim_{n\to\infty} n^{2n}/(n+1)^{n+1} = \infty$.

3.2.35 **Halley's method** for solving equation $f(x) = 0$ uses the iteration formula

$$x_{n+1} = x_n - \frac{f_n f_n'}{(f')^2 - f_n f_n''/2}$$

where $f_n = f(x_n)$ and so on. Show that this formula results when Newton's iteration is applied to $f/\sqrt{f'}$.

Proof. This follows simply from direct computation. For clarity I will denote f_n as $f(x_n)$ instead.

$$\begin{aligned}\frac{f(x_n)/\sqrt{f'(x_n)}}{\frac{\mathrm{d}}{\mathrm{d}x}[f(x_n)/\sqrt{f'(x_n)}]} &= \frac{f(x_n)}{\sqrt{f'(x_n)}}\cdot\frac{f'(x_n)}{\sqrt{f'(x_n)}f'(x_n) - f(x_n)f''(x_n)/2\sqrt{f'(x_n)}}\\ &= \frac{f(x_n)f'(x_n)}{f'(x_n)^2 - f(x_n)f''(x_n)/2}\end{aligned}$$

as desired, where the huge fraction comes from quotient rule. □

3.3.1 Establish equation (4) on page 77.

Solution

$$\begin{aligned}e_{n+1} &= \frac{f(x_n)e_{n-1} - f(x_{n-1})e_n}{f(x_n) - f(x_{n-1})} \\ &= e_n e_{n-1} \frac{f(x_n)/e_n - f(x_{n-1})e_{n-1}}{f(x_n) - f(x_{n-1})} \\ &= e_n e_{n-1} \left[\frac{x_n - x_{n-1}}{f(x_n - f(x_{n-1})}\right]\left[\frac{f(x_n)/e_n - f(x_{n-1})/e_{n-1}}{x_n - x_{n-1}}\right].\end{aligned}$$

3.3.2 In the secant method, prove that if $x_n \to q$ as $n \to \infty$ and if $f'(q) \neq 0$ then q is a zero of f.

Proof. Suppose $\{x_n\} \to q$. In particular this sequence is Cauchy, so by construction

$$f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] \to 0 \text{ as } n \to \infty.$$

Since $f'(q) = \lim_{n\to\infty} \dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \neq 0$, it must be the case that $\lim_{n\to\infty} f(x_n) \to 0$. Since $\{x_n\} \to q$ we conclude that $f(q) = 0$, as desired. □

3.3.10 The relation of asymptotic equality between two sequences is written $x_n \sim y_n$ and signifies that $\lim_{n\to\infty} x_n/y_n = 1$. Prove that if $x_n \sim y_n, u_n \sim v_n$, and $c \neq 0$, then

$$(a) c x_n \sim c y_n \qquad (b) x_n^c \sim y_n^c \qquad (c) x_n u_n \sim y_n v_n \qquad (d) y_n \sim u_n \Rightarrow x_n \sim v_n \qquad (e) y_n \sim x_n$$

Proof. Recall the important facts that if $\lim a_n = a$, $\lim b_n = b$, and $c \neq 0$ then

$$\lim a_n \lim b_n = ab \text{ and } \lambda \lim a_n = \lambda a.$$

(a) is immediate since $(cx_n)/(cy_n) = x_n/y_n$. (b) holds because $x_n^c/y_n^c = (x_n/y_n)^c$, and it remains to apply the first equation above. (c) is also obvious once we realize that $(x_n u_n)/(y_n v_n) = (x_n/y_n)(u_n/v_n)$, and once again the limit of product is the product of limits. (d) holds because

$$\lim_{n\to\infty} \frac{x_n}{v_n} = \lim_{n\to\infty} \frac{x_n}{y_n}\frac{y_n}{u_n}\frac{u_n}{v_n} = 1 \cdot 1 \cdot 1 = 1.$$

Finally, (e) holds because $\lim y_n/x_n = \lim 1/(x_n/y_n) = 1$. □

3.3.13 Prove that if V_i is defined recursively by

$$V_i = V_{i-1}(1+r) + a_i$$

for $i \geqslant 2$ with $V_1 = a_1$, then $V_n = \sum_{i=1}^{n} a_i(1+r)^{n-i}$.

Proof. We will prove this by induction. Let $\varphi(m)$ be the claim that $V_k = \sum_{i=1}^{m} a_i(1+r)^{n-i}$. Clearly $\varphi(1)$ is true as $V_1 = a_1$. Now for the inductive step, assume $\varphi(k)$ holds. Then,

$$\begin{aligned} V_{k+1} &= V_k(1+r) + a_{k+1} \\ &= (1+r)\sum_{i=1}^{k} a_i(1+r)^{k-i} + a_{k+1} \\ &= \sum_{i=1}^{k} a_i(1+r)^{k+1-i} + a_{k+1} \\ &= \sum_{i=1}^{k+1} a_i(1+r)^{k+1-i}, \end{aligned}$$

and indeed $\varphi(m)$ holds for all m. □