MATH 501 Homework 6

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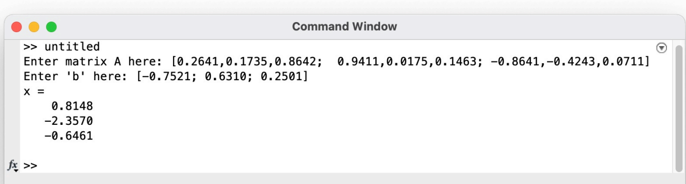
Pseudocode Implementation

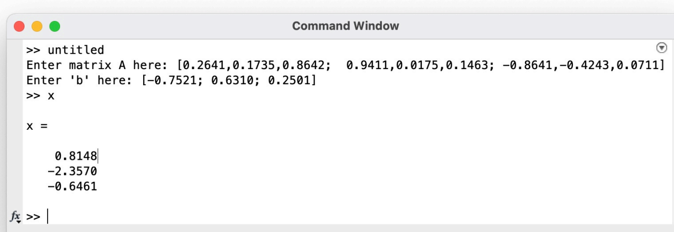
```
%Naive Gaussian

A = input("Enter matrix A here: ");
b = input("Enter b here: ");
n = size(A);
n = n(1);

for k = 1:n-1
   for i = k+1:n
       z = A(i,k) / A(k,k);
       A(i,k) = 0;
       b(i) = b(i) - z * b(k);
       for j = k+1:n
          A(i,j) = A(i,j) - z * A(k,j);
       end
   end
end

%Bwd substitution
x = zeros(n,1);

for i = 1:n
    inew = n+1-i;
    x(inew) = b(inew) - A(inew,inew+1:n) *
         x(inew+1:n);
    x(inew) = x(inew) / A(inew,inew);
end

disp("x = ");
disp(x);
```

Output of problem 4.3.22:




```
%Gaussian with scaled pivoting

A = input("Enter matrix A here: ");
b = input("Enter b here: ");
n = size(A);
n = n(1);
p = zeros(n,1);
s = zeros(n,1);
for i = 1:n
    p(i) = i;
    s(i) = max(abs(A(i,1:n)));
end

for k = 1:n-1
    [M,I] = max( abs(A(p(k:n),k)) ./ s(p(k:n)) );
    j = I + k-1;
    [p(k),p(j)] = swap(p(k),p(j));
    for i = k+1:n
        z = A(p(i),k) / A(p(k),k);
        A(p(i),k) = z;
        for l = k+1:n
           A(p(i),l) = A(p(i),l) - z * A(p(k),l);
        end
        b(p(i)) = b(p(i)) - A(p(i),k) * b(p(k));
    end
end

x = zeros(n,1);
for i = 1:n
    inew = n+1-i;
    x(inew) = b(p(inew)) - A(p(inew),inew+1:n) *
         x(inew+1:n);
    x(inew) = x(inew) / A(p(inew),inew);
end

disp("x = ");
disp(x);

function [b, a] = swap(a, b)
end
```

Output of problem 4.3.22:



Textbook Problems

4.3.1 (a) Naive Gaussian elimination:

$$\begin{bmatrix}-1 & 1 & -4\\ 2 & 2 & 0\\ 3 & 3 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 0 & 0\end{bmatrix}\begin{bmatrix}-1 & 1 & -4\\ 0 & 4 & -8\\ 0 & 6 & -10\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -3 & 1.5 & 1\end{bmatrix}\begin{bmatrix}-1 & 1 & 4\\ 0 & 4 & -8\\ 0 & 0 & 2\end{bmatrix}.$$

Solving $\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 3 & 1.5 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 0.5\end{bmatrix}$ gives $z_1 = 0, z_2 = 1$, and $z_3 = -1$. Then

$$\begin{bmatrix}-1 & 1 & 4\\ 0 & 4 & -8\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 1\\ -1\end{bmatrix} \implies \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1.25\\ -0.75\\ -0.5\end{bmatrix}$$

Pivoted Gaussian elimination:

$$\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 1 & -4\\ 2 & 2 & 0\\ 3 & 3 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -0.5 & 1 & 0\\ 1.5 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 2 & 0\\ 0 & 2 & -4\\ 0 & 0 & 2\end{bmatrix}$$

To solve $Ax = b$, write $A = P^{-1}LU$. We solve $LUx = Pb$.

$$\begin{bmatrix}1 & 0 & 0\\ -0.5 & 1 & 0\\ 1.5 & 0 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 0.5\end{bmatrix} \implies \begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}1\\ 0.5\\ -1\end{bmatrix}.$$

Then, $Ux = z$ gives

$$\begin{bmatrix}2 & 2 & 0\\ 0 & 2 & -4\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1\\ 0.5\\ -1\end{bmatrix} \implies \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}1.25\\ -0.75\\ -0.5\end{bmatrix}.$$

(b) Naive Gaussian elimination:

$$\begin{bmatrix}1 & 6 & 0\\ 2 & 1 & 0\\ 0 & 2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 6 & 0\\ 0 & -11 & 0\\ 0 & 2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & -2/11 & 1\end{bmatrix}\begin{bmatrix}1 & 6 & 0\\ 0 & -11 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

To solve the original system, we begin by solving $\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & -2/11 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}3\\ 1\\ 1\end{bmatrix} \implies \begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} = \begin{bmatrix}3\\ -5\\ 1/11\end{bmatrix}$. Then,

$$\begin{bmatrix}1 & 6 & 0\\ 0 & -11 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}3\\ -5\\ 1/11\end{bmatrix} \implies \begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}3/11\\ 5/11\\ 1/11\end{bmatrix}.$$

Pivoted Gaussian elimination:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 11/2 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 4/11 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 11/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Again we begin by solving $\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 4/11 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \implies \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/2 \\ 1/11 \end{bmatrix}$. Then,

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 11/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/2 \\ 1/11 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/11 \\ 5/11 \\ 1/11 \end{bmatrix}.$$

4.3.3 Let $\sigma = (p_1, p_2, \ldots, p_n)$ be a permutation.

(1) PA permutes the rows of A according to σ, i.e., the n^{th} row of A becomes the p_n^{th} of PA.

(2) $AP = (P^T A^T)^T = (P^{-1} A^T)^T$ permutes the columns of A according to σ^{-1} (since $P^T = P^{-1}$, as shown below), i.e., the p_i^{th} column of A becomes the n^{th} of AP.

(3) P^{-1} is simply P^T: $(PP^T)_{i,j} = \sum_{k=1}^{n} P_{i,k} P^T_{k,j} = \sum_{k=1}^{n} P_{i,k} P_{j,k} = \delta_{i,j}$ (the Kronecker δ).

(4) PA first permutes the rows of A according to σ. Then the columns of PA is permuted according to σ by P^{-1} (since right multiplication of P permutes the columns by σ^{-1}). Hence $(PAP^{-1})_{i,j} = A_{p_i, p_j}$.

4.3.14 Let A be an $n \times n$ tridiagonal matrix. To solve $Ax = b$, we first need to eliminate all the subdiagonal entries. As suggested by the text, eliminating one entry requires 3 ops: one to determine the factor, one to determine how the corresponding diagonal entry changes, and one to determine how the corresponding entry in b changes. After these $3(n-1)$ ops, we are left with a bidiagonal, upper-triangular matrix and $\tilde{A}x = \tilde{b}$. Performing back substitution, for each x_n we need to calculate the product of the superdiagonal entry $(A_{n,n+1})$ with x_{n+1} (which we already know by the time we get to x_n) and factorized quotient $(\tilde{b}_n - A_{n,n+1}x_{n+1})/A_{n,n}$, hence another $2(n-1)$ ops. To sum up, we require approximately $3(n-1)$ ops in elimination phase and $2(n-1)$ in back substitution phrase, which add up a total of $5(n-1) = \mathcal{O}(n)$ ops.

4.3.20 We want to solve $\begin{bmatrix} \epsilon & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. Eliminating the bottom-left entry gives

$$\begin{bmatrix} \epsilon & 2 \\ 0 & -1 - 2/\epsilon \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 - 4/\epsilon \end{bmatrix}.$$

`MARC-32` will first try to solve the equation $x_2 = \frac{-1 - 2/\epsilon}{-1 - 4/\epsilon}$. If $\epsilon < 2^{-22}$ then $2/\epsilon > 2^{23}$ and $4/\epsilon > 2^{24}$. The original 1's will no longer appear in the mantissa and thus $x_2 = 1/2$ and $x_1 = (4 - 2x_2)/\epsilon > 3 \cdot 2^{23}$, causing an overflow.

4.3.33 Notice that if $p : J \to J$ is subjective the it must be injective, since otherwise $a \neq b$ but $p(a) = p(b)$ implies that p has to map the remaining $n-2$ elements in the domain to $n-1$ elements in the codomain, which is absurd. Thus a permutation is not only a surjection but also a bijection.

Clearly, if $p, q : J \to J$ are bijections, then so is their composition, and thus $p \circ q$ is another permutation. The associativity is guaranteed as part of the properties of function compositions. For the identity, for all $1 \leqslant x \leqslant n$ we have

$$(p \circ u)(x) = p(u(x)) = p(x) = u(p(x)) = (u \circ p)(x).$$

4.3.34 Let p be an arbitrary permutation and define $p_i := p(i)$. By above p is bijective and thus admits an inverse, which we call p^{-1}. Then the properties $p \circ p^{-1} = u = p^{-1} \circ p$ is immediate by the definition of an inverse of a bijective map.

4.3.56 It is true! Since one can inductively show the remaining steps after showing the preservation of diagonal dominance from $A^{(1)}$ to $A^{(2)}$, it suffices to show how it is preserved in the first step, namely

$$\left| a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} \right| \geqslant \sum_{i=3}^{n} \left| a_{2,i} - a_{1,i} \cdot \frac{a_{21}}{a_{11}} \right|,$$

which, by multiplying a_{11} on both sides, is equivalent to showing

$$|a_{22}a_{11} - a_{12}a_{21}| \geqslant \sum_{i=3}^{n} |a_{2,i}a_{11} - a_{1,i}a_{21}|.$$

Indeed,

$$\begin{aligned}
\sum_{i=3}^{n} |a_{2,i}a_{11} - a_{1,i}a_{21}| &\leqslant \sum_{i=3}^{n} |a_{2,i}a_{11}| + \sum_{i=3}^{n} |a_{1,i}a_{21}| \\
&\leqslant |a_{11}| \sum_{i=3}^{n} |a_{2,i}| + |a_{21}| \sum_{i=3}^{n} |a_{1,i}| \\
&\leqslant |a_{11}|(|a_{22}| - |a_{21}|) + |a_{21}|(|a_{11}| - |a_{12}|) \\
&= |a_{11}||a_{22}| - |a_{21}||a_{12}| \\
&= |a_{11}a_{22}| + |a_{21}a_{12}| \\
&\leqslant |a_{11}a_{22} - a_{21}a_{12}|.
\end{aligned}$$

Therefore the claim follows. □

4.3.57

(a) I believe $(n-1)(n!)$ correspond to the "big formula", not expansion by minors. Anyway, there are $n!$ ways to permute $\{1, \ldots, n\}$ to $\{p_1, \ldots, p_n\}$, and then when calculating the product $a_{1,p_1}a_{2,p_2} \ldots a_{n,p_n}$ there are $n-1$ multiplications involved. Therefore $n!$ terms, each with $n-1$ ops, give us a total of $(n-1)(n!)$ ops.

(b) Besides computing the determinant of A, we need to compute n more determinants, each corresponding to one component of x, hence the total $(n+1)(n-1)(n!) = (n^2-1)(n!)$ ops.

(c) To carry out the Gauss-Jordan elimination, we first write the matrix in augmented form $\left[A \mid b \right]$. Then we scale the first row so that $a_{11} = 1$, after which we eliminate other entries of form $a_{1,i}$. After we are done with the $(n-1)^{\text{th}}$ column we proceed to set $a_{n,n} = 1$ and eliminate other entries on the n^{th} column.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \end{array}\right] \xrightarrow{(n-1)+1 \text{ ops}} \left[\begin{array}{cccc|c} 1 & a'_{12} & \cdots & a_{1n} & b'_1 \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \end{array}\right]$$

$$\xrightarrow{n(n-1) \text{ ops}} \left[\begin{array}{cccc|c} 1 & * & \cdots & * & * \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} & b'_n \end{array}\right]$$

$$\vdots$$

$$\xrightarrow{(n-n)+1 \text{ ops}} \left[\begin{array}{cccc|c} 1 & 0 & \cdots & * & b'_1 \\ 0 & 1 & \cdots & * & b'_2 \\ \vdots & \vdots & \ddots & * & \vdots \\ 0 & 0 & \cdots & 1 & b'_n \end{array}\right]$$

$$\xrightarrow{1\cdot(n-1) \text{ ops}} \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \tilde{b}_1 \\ 0 & 1 & \cdots & 0 & \tilde{b}_2 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & \tilde{b}_n \end{array}\right].$$

When dealing with the n^{th} column, we first need $n-1+1$ divisions to scale n^{th} row (setting $a_{nn} = 1$ saves one *ops* but the augmented b_n requires an extra *ops*). Then we eliminate all other entries in the n^{th} column (as marked red above) and compute $n(n-1)$ *ops* to all the remaining nonzero entries. The total amount of *ops* required by doing so is

$$\sum_{k=n}^{1} k + \sum_{k=n}^{1} k(n-1) = \frac{n^2(n+1)}{2} \approx \frac{n^3}{2},$$

indeed 50% more expensive than Gaussian elimination's $n^3/3$.