MATH 501 Homework 7

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4.4.1 Show that the norms $\|x\|_\infty, \|x\|_2, \|x\|_1$ satisfy the postulates that define a norm.

Proof. Assuming the vector space is $\mathbb{R}^n$ (since proof for $\mathbb{C}^n$ or ℓ^p spaces will be slightly different). Non-degeneracy of all three norms are all trivial. Absolute homogeneity follows from the following:

$$\|\lambda x\|_1 = \sum_{i=1}^n |\lambda x_i| = \lambda \sum_{i=1}^n |x_i| = \lambda \|x\|_1,$$

$$\|\lambda x\|_2 = \left(\sum_{i=1}^n |\lambda x_i|^2\right)^{1/2} = \left(\lambda^2 \sum_{i=1}^n |x_i|^2\right)^{1/2} = \lambda \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \lambda \|x\|_{1/2}, \text{ and}$$

$$\|\lambda x\|_\infty = \max_{1 \leqslant i \leqslant n} |\lambda x_i| = \lambda \max_{1 \leqslant i \leqslant n} |x_i| = \lambda \|x\|_\infty.$$

For subadditivity of $\|\cdot\|_1$, notice that

$$\|u\|_1 + \|v\|_1 = \sum_{i=1}^n |u_i + v_i| \geqslant \sum_{i=1}^n (|u_i| + |v_i|) = \sum_{i=1}^n |u_i| + \sum_{i=1}^n |v_i| = \|u\|_1 + \|v\|_1.$$

For subadditivity of $\|\cdot\|_2$, it suffices to show $(\|u\|_2 + \|v\|_2)^2 \geqslant \|u + v\|_2^2$. Indeed,

$$\|u + v\|_2^2 = \sum_{i=1}^n |u_i + v_i|^2 = \sum_{i=1}^n |u_i|^2 + \sum_{i=1}^n |v_i|^2 + \sum_{i=1}^n 2|u_i||v_i|$$

$$= \|u\|_2^2 + \|v\|_2^2 + 2\sum_{i=1}^n |u_i||v_i|$$

$$\leqslant \|u\|_2^2 + \|v\|_2^2 + 2\sqrt{\sum_{i=1}^n |u_i|^2}\sqrt{\sum_{i=1}^n |v_i|^2}$$

$$= (\|u\|_2 + \|v\|_2)^2.$$

The red step comes from Cauchy-Scuwarz inequality, which requires a nontrivial proof.

Proof of Cauchy-Schwarz. We begin by defining an inner product $\langle\cdot,\cdot\rangle:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_{\geqslant 0}$ by

$$\langle x,y\rangle := \sum_{i=1}^n x_i y_i.$$

Notice that $\langle\cdot,\cdot\rangle$ is linear with respect to either argument, i.e.,

$$\langle x,y_1+\lambda y_2\rangle = \langle x,y_1\rangle + \lambda\langle x,y_2\rangle \ \text{ and } \ \langle x_1+\lambda x_2,y\rangle = \langle x_1,y\rangle + \lambda\langle x_2,y\rangle$$

and $\langle x,y\rangle = \langle y,x\rangle$. Also notice that our previously defined $\|x\|_2 = \sqrt{\langle x,x\rangle}$. (Indeed, $\|\cdot\|_2$ is induced by the inner product.) Notice that $\|x\|_2 \geqslant 0$ implies $\langle x,x\rangle \geqslant 0$ for all $x\in\mathbb{R}^n$, including $u+\lambda v$ for $\lambda\in\mathbb{R}$. Thus,

$$\langle u+\lambda v,u+\lambda v\rangle = \lambda^2\langle v,v\rangle + 2\lambda\langle u,v\rangle + \langle u,u\rangle,$$

a quadratic polynomial of λ that has a nonpositive discriminant (since $\langle\cdot,\cdot\rangle \geqslant 0$ and it can have at most one distinct root). Therefore,

$$(2\langle u,v\rangle)^2 - 4\cdot\langle u,u\rangle\langle v,v\rangle \leqslant 0 \implies \langle u,v\rangle^2 \leqslant \|u\|^2\|v\|^2 \implies \langle u,v\rangle \leqslant \|u\|\|v\|.$$

We've therefore proven Cauchy-Schwarz inequality for norms over $\mathbb{R}$. $\qquad\square$

For subadditivity of $\|\cdot\|_\infty$, we have

$$\|u+v\|_\infty = \max_{1\leqslant i\leqslant n}|u_i+v_i| \leqslant \max_{1\leqslant i\leqslant n}\left(|u_i|+|v_i|\right) = \max_{1\leqslant i\leqslant n}|u_i| + \max_{1\leqslant i\leqslant n}|v_i| = \|u\|_\infty + \|v\|_\infty.$$

Therefore $\|\cdot\|_\infty, \|\cdot\|_1$, and $\|\cdot\|_2$ are all well-defined norms. $\qquad\square$

4.4.2 Show that $\|x\|_\infty \leqslant \|x\|_2 \leqslant \|x\|_1$ for $x\in\mathbb{R}^n$ and that there are nontrivial examples that attain the equalities.

Proof. Clearly $\|x\|_\infty \leqslant \|x\|_p$ for any p, as

$$\|x\|_\infty^p = \max_{1\leqslant i\leqslant n}|x_i|^p \leqslant \sum_{i=1}^n|x_i|^p = \|x\|_p^p.$$

Instead of showing $\|x\|_2 \leqslant \|x\|_1$, I'd like to show $\|x\|_q \leqslant \|x\|_p$ whenever $p\leqslant q$.

(1) If $\|x\|_p = 1$, then $\sum_{i=1}^n|x_i|^p = 1$ which implies each individual $|x_i|^p \leqslant 1$, and so $|x_i|\leqslant 1$. Then,

$$\|x\|_q = \sum_{i=1}^n|x_i|^q \leqslant \sum_{i=1}^n|x_i|^p = 1.$$

(2) If $\|x\|_p \neq 1$, we can first normalize it to $y := x/\|x\|_p$ so that $\|y\| = 1$. Then,

$$\|x\|_q = \|\underbrace{\left(\|x\|_p\right)}_{\in\mathbb{R}_+} y\|_q = \|x\|_p\|y\|_q \leqslant \|x\|_p \text{ by (1)}.$$

Therefore $\|x\|_q \leqslant \|x\|_p$ for all $p\leqslant q$, and of course $\|x\|_2 \leqslant \|x\|_1$ is just one special case. The equalities can be easily obtained if we set x to be any standard basis for $\mathbb{R}^n$, for example $(1,0,\dots)$, in which case all three norms evaluate to 1. $\qquad\square$

4.4.3 Show that $\|x\|_1 \leqslant n\|x\|_\infty$ and $\|x\|_2 \leqslant \sqrt{n}\|x\|_\infty$ for $x\in\mathbb{R}^n$.

Proof. The first one is immediate since

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leqslant n \left(\max_{1 \leqslant i \leqslant n} |x_i| \right) = n\|x\|_\infty.$$

The second one is analogous. Since both sides are nonnegative it suffices to prove $\|x\|_2^2 \leqslant n\|x\|_\infty^2$. Indeed,

$$\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2 \leqslant n \left(\max_{1 \leqslant i \leqslant n} |x_i|^2 \right) = n\|x\|_\infty^2. \qquad \square$$