MATH 501 Homework 7

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4.4.1 Show that the norms $\|x\|_\infty, \|x\|_2, \|x\|_1$ satisfy the postulates that define a norm.

Proof. Assuming the vector space is $\mathbb{R}^n$ (since proof for $\mathbb{C}^n$ or ℓ^p spaces will be slightly different). Non-degeneracy of all three norms are all trivial. Absolute homogeneity follows from the following:

$$\|\lambda x\|_1 = \sum_{i=1}^n |\lambda x_i| = \lambda \sum_{i=1}^n |x_i| = \lambda \|x\|_1,$$

$$\|\lambda x\|_2 = \left(\sum_{i=1}^n |\lambda x_i|^2\right)^{1/2} = \left(\lambda^2 \sum_{i=1}^n |x_i|^2\right)^{1/2} = \lambda \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} = \lambda \|x\|_{1/2}, \text{ and}$$

$$\|\lambda x\|_\infty = \max_{1\leqslant i\leqslant n} |\lambda x_i| = \lambda \max_{1\leqslant i\leqslant n} |x_i| = \lambda \|x\|_\infty.$$

For subadditivity of $\|\cdot\|_1$, notice that

$$\|u\|_1 + \|v\|_1 = \sum_{i=1}^n |u_i + v_i| \geqslant \sum_{i=1}^n (|u_i| + |v_i|) = \sum_{i=1}^n |u_i| + \sum_{i=1}^n |v_i| = \|u\|_1 + \|v\|_1.$$

For subadditivity of $\|\cdot\|_2$, it suffices to show $(\|u\|_2 + \|v\|_2)^2 \geqslant \|u+v\|_2^2$. Indeed,

$$\begin{aligned} \|u+v\|_2^2 &= \sum_{i=1}^n |u_i + v_i|^2 = \sum_{i=1}^n |u_i|^2 + \sum_{i=1}^n |v_i|^2 + \sum_{i=1}^n 2|u_i||v_i| \\ &= \|u\|_2^2 + \|v\|_2^2 + 2\sum_{i=1}^n |u_i||v_i| \\ &\leqslant \|u\|_2^2 + \|v\|_2^2 + 2\sqrt{\textstyle\sum_{i=1}^n |u_i|^2}\sqrt{\textstyle\sum_{i=1}^n |v_i|^2} \\ &= (\|u\|_2 + \|v\|_2)^2. \end{aligned}$$

The red step comes from Cauchy-Scuwarz inequality, which requires a nontrivial proof. A proof of Cauchy-Schwarz has been included on the next page.

For subadditivity of $\|\cdot\|_\infty$, we have

$$\|u+v\|_\infty = \max_{1\leqslant i\leqslant n} |u_i + v_i| \leqslant \max_{1\leqslant i\leqslant n} (|u_i| + |v_i|) = \max_{1\leqslant i\leqslant n} |u_i| + \max_{1\leqslant i\leqslant n} |v_i| = \|u\|_\infty + \|v\|_\infty.$$

Therefore $\|\cdot\|_\infty, \|\cdot\|_1$, and $\|\cdot\|_2$ are all well-defined norms. ☐

Proof of Cauchy-Schwarz. We begin by defining an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ by

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i.$$

Notice that $\langle \cdot, \cdot \rangle$ is linear with respect to either argument, i.e.,

$$\langle x, y_1 + \lambda y_2 \rangle = \langle x, y_1 \rangle + \lambda \langle x, y_2 \rangle \text{ and } \langle x_1 + \lambda x_2, y \rangle = \langle x_1, y \rangle + \lambda \langle x_2, y \rangle$$

and $\langle x, y \rangle = \langle y, x \rangle$. Also notice that our previously defined $\|x\|_2 = \sqrt{\langle x, x \rangle}$. (Indeed, $\| \cdot \|_2$ is induced by the inner product.) Notice that $\|x\|_2 \geqslant 0$ implies $\langle x, x \rangle \geqslant 0$ for all $x \in \mathbb{R}^n$, including $u + \lambda v$ for $\lambda \in \mathbb{R}$. Thus,

$$\langle u + \lambda v, u + \lambda v \rangle = \lambda^2 \langle v, v \rangle + 2\lambda \langle u, v \rangle + \langle u, u \rangle,$$

a quadratic polynomial of λ that has a nonpositive discriminant (since $\langle \cdot, \cdot \rangle \geqslant 0$ and it can have at most one distinct root). Therefore,

$$(2 \langle u, v \rangle)^2 - 4 \cdot \langle u, u \rangle \langle v, v \rangle \leqslant 0 \implies \langle u, v \rangle^2 \leqslant \|u\|^2 \|v\|^2 \implies \langle u, v \rangle \leqslant \|u\| \|v\|.$$

We've therefore proven Cauchy-Schwarz inequality for norms over $\mathbb{R}$. □

4.4.2 Show that $\|x\|_\infty \leqslant \|x\|_2 \leqslant \|x\|_1$ for $x \in \mathbb{R}^n$ and that there are nontrivial examples that attain the equalities.

Proof. Clearly $\|x\|_\infty \leqslant \|x\|_p$ for any p, as

$$\|x\|_\infty^p = \max_{1 \leqslant i \leqslant n} |x_i|^p \leqslant \sum_{i=1}^{n} |x_i|^p = \|x\|_p^p.$$

Instead of showing $\|x\|_2 \leqslant \|x\|_1$, I'd like to show $\|x\|_q \leqslant \|x\|_p$ whenever $p \leqslant q$.

(1) If $\|x\|_p = 1$, then $\sum_{i=1}^{n} |x_i|^p = 1$ which implies each individual $|x_i|^p \leqslant 1$, and so $|x_i| \leqslant 1$. Then,

$$\|x\|_q = \sum_{i=1}^{n} |x_i|^q \leqslant \sum_{i=1}^{n} |x_i|^p = 1.$$

(2) If $\|x\|_p \neq 1$, we can first normalize it to $y := x / \|x\|_p$ so that $\|y\| = 1$. Then,

$$\|x\|_q = \| \underbrace{(\|x\|_p)}_{\in \mathbb{R}_+} y \|_q = \|x\|_p \|y\|_q \leqslant \|x\|_p \text{ by (1).}$$

Therefore $\|x\|_q \leqslant \|x\|_p$ for all $p \leqslant q$, and of course $\|x\|_2 \leqslant \|x\|_1$ is just one special case. The equalities can be easily obtained if we set x to be any standard basis for $\mathbb{R}^n$, for example $(1, 0, \dots)$, in which case all three norms evaluate to 1. □

4.4.3 Show that $\|x\|_1 \leqslant n \|x\|_\infty$ and $\|x\|_2 \leqslant \sqrt{n} \|x\|_\infty$ for $x \in \mathbb{R}^n$.

Proof. The first one is immediate since

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \leqslant n \left(\max_{1 \leqslant i \leqslant n} |x_i| \right) = n \|x\|_\infty.$$

The second one is analogous. Since both sides are nonnegative it suffices to prove $\|x\|_2^2 \leqslant n\|x\|_\infty^2$. Indeed,

$$\|x\|_2^2 = \sum_{i=1}^n |x_i|^2 \leqslant n\left(\max_{1\leqslant i\leqslant n}|x_i|^2\right) = n\|x\|_\infty^2. \qquad \square$$

4.4.7 Determine if the following define norms on $\mathbb{R}^n$.

(a) $\max\{|x_2|,\dots,|x_n|\}$. **No**. $(1,0,\dots) \neq 0$ but this expression evaluates to 0.

(b) $\sum_{i=1}^n |x_i|^3$. **No**: subaddtivity is not satisfied. Consider $(1,\dots,1)$ which evaluates to n. $(2,\dots,2)$, however, evaluates to $8n > n+n$.

(c) $\left(\sum_{i=1}^n \sqrt{|x_i|}\right)^2$. **No**. Consider $(0,1),(1,1) \in \mathbb{R}^2$. Then

$$\begin{cases}(4,0) \mapsto (2+0)^2 = 4 \\ (0,4) \mapsto (0+2)^2 = 4\end{cases} \quad \text{but } (4,0)+(0,4) = (4,4) \mapsto (2+2)^2 = 16 > 4+4.$$

(d) $\max\{|x_1-x_2|,|x_1+x_2|,|x_3|,|x_4|,\dots,|x_n|\}$. **Yes, this is a norm**. Define $f:\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ by $f(x) \coloneqq$ the max function described in this problem. Clearly the codomain is $R_{\geqslant 0}$ by construction.

(1) Non-degeneracy: $f(x) \geqslant 0$ since absolute values are nonnegative. If $f(x) = 0$ then $x_3 = \cdots = x_n = 0$ and $|x_1 - x_2| = |x_1 + x_2| = 0$. This means $x_1 = x_2 = 0$ as well and thus $x = 0$.

(2) Absolute homogeneity: this follows directly from the fact that $|\lambda v| = |\lambda||v|$.

(3) Subadditivity:

$$\begin{aligned} f(x+y) &= \max\{|x_1+y_1-x_2-y_2|,|x_1+y_1+x_2+y_2|,|x_3+y_3|,\dots,|x_n+y_n|\} \\ &\leqslant \max\{|x_1-x_2|+|y_1-y_2|,|x_1+x_2|+|y_1+y_2|,|x_3|+|y_3|,\dots\} \\ &\leqslant \max\{|x_1-x_2|,|x_1+x_2|,|x_3|,\dots\} + \max\{|y_1-y_2|,|y_1+y_2|,|y_3|,\dots\} \\ &= f(x)+f(y). \end{aligned}$$

Therefore, weird as it sounds, f actually defines a norm on $\mathbb{R}^n$.

(e) $\sum_{i=1}^n 2^{-i}|x_i|$. **This is yet another norm**. Like above, non-degeneracy and absolute homogeneity are clear. It remains to show aubadditivity. Indeed,

$$\begin{aligned} \sum_{i=1}^n 2^{-i}|x_i+y_i| &\leqslant \sum_{i=1}^n 2^{-i}(|x_i|+|y_i|) \\ &= \sum_{i=1}^n 2^{-i}|x_i| + \sum_{i=1}^n 2^{-i}|y_i|. \end{aligned}$$

4.4.8 Define $\|A\| \coloneqq \sum_{i=1}^n\sum_{j=1}^n |a_{ij}|$. Show that this is a matrix norm. Show that it is not subordinate to any vector norm. Does it satisfy $\|I\| = 1$ and $\|AB\| \leqslant \|A\|\|B\|$?

Proof. Notice that $\|A\|$ is very similar to the $\|\cdot\|_1$ on $\mathbb{R}^{n\times n}$. Therefore all three postulate immediately follow. However, $\|A\|$ is not subordnated, as $\|I\| = n > 1$ for $n > 1$. It does, however, satisfy (10). If A and B have only nonnegative entries, then

$$\|AB\| = \sum_{i=1}^n\sum_{j=1}^n |(ab)_{ij}| = \sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n a_{ik}b_{kj} = \sum_{i=1}^n\left(\sum_{j=1}^n a_{ij}\|B\|\right) = \|A\|\|B\|.$$

On the other hand, if there is any negative entry in the i^{th} row of A or the j^{th} column of B, then

$$|(ab)_{ij}| = \left|\sum_{k=1}^{n} a_{ik}b_{kj}\right| \leqslant \sum_{k=1}^{n}|a_{ik}b_{kj}| = \sum_{k=1}^{n}|a_{ik}||b_{kj}|,$$

and so in this case $\|AB\| \leqslant \|A\|\|B\|$. Either way, (10) holds. □

4.4.11 Show that the matrix norm $\|A\|_1$ subordinate to $\|x\|_1$ ($x \in \mathbb{R}^n$) is

$$\|A\|_1 = \max_{1\leqslant j\leqslant n}\sum_{i=1}^{n}|a_{ij}|.$$

Proof. Let A be given. Let the $\tilde{j}^{\text{th}}$ column be the one with largest 1-norm. Let A_i denote the i^{th} column of A. Then,

$$\begin{aligned}\|A\|_1 = \sup_{\|x\|_1=1}\|Ax\|_1 &= \sup_{\|x\|_1=1}\left\|\sum_{i=1}^{n}x_iA_i\right\|_1 \\ &\leqslant \sup_{\|x\|_1=1}\sum_{i=1}^{n}\|x_iA_i\|_1 = \sup_{\|x\|_1=1}\sum_{i=1}^{n}|x_i|\|A_i\|_1 \\ &\leqslant \sup_{\|x\|_1=1}\sum_{i=1}^{n}|x_i|\|A_{\tilde{j}}\|_1 = \|A_{\tilde{j}}\|_1 \sup_{\|x\|_1=1}\sum_{i=1}^{n}|x_i| \\ &= \|A_{\tilde{j}}\|_1.\end{aligned}$$

The other direction is much easier: simply consider $\tilde{x} \in \mathbb{R}^n$ with $\tilde{x}_i = \delta_{i,\tilde{j}}$, i.e., 1 for the $\tilde{j}^{\text{th}}$ entry and 0 otherwise. Then $\|A\tilde{x}\|_1 = \|A_{\tilde{j}}\|_1$ and the $\geqslant$ follows from definition of supremum. □

4.4.23 Prove that if $\|\cdot\|$ is a norm on a vector space, and if we define $\|x\|' \coloneqq \alpha\|x\|$ with a fixed $\alpha > 0$, then $\|\cdot\|'$ also defines a norm.

Proof. Non-degeneracy is trivial. So is absolute homogeneity since $\|\lambda x\|' = \alpha\|\lambda x\| = |\lambda|\alpha\|x\| = |\lambda|\|x\|'$. For triangle inequality,

$$\|x + y\|' = \alpha\|x + y\| \leqslant \alpha(\|x\| + \|y\|) = \alpha\|x\| + \alpha\|y\| = \|x\|' + \|y\|'.$$ □

4.4.24 If the construction in the preceeding problem is applied to a subordinate matrix norm, is the resulting norm also a subordinate matrix norm?

Solution

Yes.

Nonnegativity is trivial. Now suppose $\|A\|' = 0$. Clearly if $A = 0$ then $\|A\|' = 0$. For the converse, if $\|A\|' = 0$ but A is not the zero matrix, then some column of A needs to be nonzero (not all 0's). Consider $v = (0, \dots, 0, 1/\alpha, 0, \dots)^T$ where the $1/\alpha$ corresponds to that column. Then $\|v\|' = 1$ and $Av \neq 0$, contradiction.

Absolute homogeneity is also clear: $\|\lambda A\|' = \sup_{\|x\|'=1}\|\lambda Ax\|' = |\lambda| \sup_{\|x\|'=1}\|Ax\|' = |\lambda|\|A\|'$.

For triangle inequality:

$$\begin{aligned}\|A+B\|' &= \sup_{\|x\|'=1} \|(A+B)x\|' = \sup_{\|x\|'=1} \|Ax+Bx\|' \\ &\leqslant \sup_{\|x\|'=1} (\|Ax\|' + \|Bx\|') \leqslant \sup_{\|x\|'=1} \|Ax\|' + \sup_{\|x\|'=1} \|Bx\|' \\ &= \|A\|' + \|B\|'.\end{aligned}$$

Therefore this new norm is still subordinated. □

If, on the other hand, the question is asking us to simply multiply everthing obtained from $\|\cdot\|$ by α, then this does **not** define a norm in general, because if this is the case, then $\|I\|' = \alpha\|I\| = \alpha = 1$ if and only if $\alpha = 1$. I am not sure which one the problem actually refers to, so I listed both.

4.4.29 Prove that if A has a nontrivial fixed point then $\|A\| \geqslant 1$ for any subordinate matrix norm.

Proof. Suppose $Ax = x$ for some $x \neq 0$. Normalizing x to $\tilde{x} := x/\|x\|$ we see that

$$A\tilde{x} = \tilde{x} \implies \|A\tilde{x}\| = \|\tilde{x}\| = 1 \implies \|A\| = \sup_{\|x\|=1} \|Ax\| \geqslant \|A\tilde{x}\| = 1.$$ □

4.4.34 For any $n \times n$ matrix A, define the **Frobenius norm** to be $\|A\|_F := \left(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$. Show that this defines a norm on the vector space of all $n \times n$ matrices. What about $\|A\| := \max_{1\leqslant i,j\leqslant n} |a_{ij}|$?

Proof (Frobenius). To see that $\|A\|_F$ defines a norm, simply notice that $\|A\|_F$ is the same as the 2-norm of a super long vector in $\mathbb{R}^{n\times n}$: $(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{nn})^T$. Therefore $\|A\|_F$ automatically satisfies all three postulates.

However, the Frobenius norm is not subordinated. For example, $Ix = x$ for all x, so if $\|\cdot\|_F$ were subordinate, one would have $\|I\|_F = \sup_{\|x\|=1} \|Ix\| = \sup_{\|x\|=1} \|x\| = 1$, while in reality $\|I\|_F = \sqrt{n}$ and n can > 1. □

Proof (max norm). Non-degeneracy and absolute homogeneity are trivial. For triangle inequality,

$$\max_{1\leqslant i,j\leqslant n} |a_{ij} + b_{ij}| \leqslant \max_{1\leqslant i,j\leqslant n} (|a_{ij}| + |b_{ij}|) \leqslant \max_{1\leqslant i,j\leqslant n} |a_{ij}| + \max_{1\leqslant \tilde{i},\tilde{j}\leqslant n} |b_{ij}|.$$

This norm, unfortunately, is not subordinated, either. For if it were, then

$$\|AB\| = \sup_{\|x\|=1} \|ABx\| \leqslant \sup_{\|x\|=1} [\|A\|\|Bx\|] \leqslant \sup_{\|x\|=1} [\|A\|\|B\|\|x\|] = \|A\|\|B\|.$$

However, notice that if we let $A = B = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$ then $AB = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$. Then $\|A\| = \|B\| = 1$ but $\|AB\| = 2$. □

4.4.37 Prove that for each $x \in \mathbb{R}^n$,

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty.$$

Remark: the same equation holds for $x \in \ell^q$ or L^q, $q \geqslant 1$.

Proof. On one hand, since each $|x_i|$ is bounded by the largest one, i.e., $\max_{1\leqslant i\leqslant n}|x_i|$,

$$\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p} \leqslant \left(n\cdot\max_{1\leqslant i\leqslant n}|x_i|^p\right)^{1/p} = n^{1/p}\max_{1\leqslant i\leqslant n}|x_i| = n^{1/p}\|x\|_\infty.$$

On the other hand, since $\sum_{i=1}^{n}$ includes the largest one and other $|x_i|$'s are also nonnegative,

$$\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p} \geqslant \left(\max_{1\leqslant i\leqslant n}|x_i|^p\right)^{1/p} = \max_{1\leqslant i\leqslant n}|x_i| = \|x\|_\infty.$$

Since $p^{1/p} = \exp((1/p)\log(p))$ and L'Hôpital's rule gives

$$\lim_{p\to\infty}\frac{\log(p)}{p} = \lim_{p\to\infty}\frac{1/p}{1} = 0,$$

we know $\lim_{p\to\infty} p^{1/p} = e^0 = 1$. Therefore, by squeeze theorem, $\lim_{p\to\infty}\|x\|_p = \|x\|_\infty$. □

End of Homework 7