

# MATH 501 Homework 7

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4.4.1 Show that the norms  $\|x\|_\infty, \|x\|_2, \|x\|_1$  satisfy the postulates that define a norm.

*Proof.* Assuming the vector space is  $\mathbb{R}^n$  (since proof for  $\mathbb{C}^n$  or  $\ell^p$  spaces will be slightly different). Non-degeneracy of all three norms are all trivial. Absolute homogeneity follows from the following:

$$\begin{aligned}\|\lambda x\|_1 &= \sum_{i=1}^n |\lambda x_i| = \lambda \sum_{i=1}^n |x_i| = \lambda \|x\|_1, \\ \|\lambda x\|_2 &= \left( \sum_{i=1}^n |\lambda x_i|^2 \right)^{1/2} = \left( \lambda^2 \sum_{i=1}^n |x_i|^2 \right)^{1/2} = \lambda \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} = \lambda \|x\|_2, \text{ and} \\ \|\lambda x\|_\infty &= \max_{1 \leq i \leq n} |\lambda x_i| = \lambda \max_{1 \leq i \leq n} |x_i| = \lambda \|x\|_\infty.\end{aligned}$$

For subadditivity of  $\|\cdot\|_1$ , notice that

$$\|u\|_1 + \|v\|_1 = \sum_{i=1}^n |u_i + v_i| \geq \sum_{i=1}^n (|u_i| + |v_i|) = \sum_{i=1}^n |u_i| + \sum_{i=1}^n |v_i| = \|u\|_1 + \|v\|_1.$$

For subadditivity of  $\|\cdot\|_2$ , it suffices to show  $(\|u\|_2 + \|v\|_2)^2 \geq \|u + v\|_2^2$ . Indeed,

$$\begin{aligned}\|u + v\|_2^2 &= \sum_{i=1}^n |u_i + v_i|^2 = \sum_{i=1}^n |u_i|^2 + \sum_{i=1}^n |v_i|^2 + \sum_{i=1}^n 2|u_i||v_i| \\ &= \|u\|_2^2 + \|v\|_2^2 + 2 \sum_{i=1}^n |u_i||v_i| \\ &\leq \|u\|_2^2 + \|v\|_2^2 + 2\sqrt{\sum_{i=1}^n |u_i|^2} \sqrt{\sum_{i=1}^n |v_i|^2} \\ &= (\|u\|_2 + \|v\|_2)^2.\end{aligned}$$

The red step comes from Cauchy-Schwarz inequality, which requires a nontrivial proof. A proof of Cauchy-Schwarz has been included on the next page.

For subadditivity of  $\|\cdot\|_\infty$ , we have

$$\|u + v\|_\infty = \max_{1 \leq i \leq n} |u_i + v_i| \leq \max_{1 \leq i \leq n} (|u_i| + |v_i|) = \max_{1 \leq i \leq n} |u_i| + \max_{1 \leq i \leq n} |v_i| = \|u\|_\infty + \|v\|_\infty.$$

Therefore  $\|\cdot\|_\infty, \|\cdot\|_1$ , and  $\|\cdot\|_2$  are all well-defined norms. □

*Proof of Cauchy-Schwarz.* We begin by defining an inner product  $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  by

$$\langle x, y \rangle := \sum_{i=1}^n x_i y_i.$$

Notice that  $\langle \cdot, \cdot \rangle$  is linear with respect to either argument, i.e.,

$$\langle x, y_1 + \lambda y_2 \rangle = \langle x, y_1 \rangle + \lambda \langle x, y_2 \rangle \quad \text{and} \quad \langle x_1 + \lambda x_2, y \rangle = \langle x_1, y \rangle + \lambda \langle x_2, y \rangle$$

and  $\langle x, y \rangle = \langle y, x \rangle$ . Also notice that our previously defined  $\|x\|_2 = \sqrt{\langle x, x \rangle}$ . (Indeed,  $\|\cdot\|_2$  is induced by the inner product.) Notice that  $\|x\|_2 \geq 0$  implies  $\langle x, x \rangle \geq 0$  for all  $x \in \mathbb{R}^n$ , including  $u + \lambda v$  for  $\lambda \in \mathbb{R}$ . Thus,

$$\langle u + \lambda v, u + \lambda v \rangle = \lambda^2 \langle v, v \rangle + 2\lambda \langle u, v \rangle + \langle u, u \rangle,$$

a quadratic polynomial of  $\lambda$  that has a nonpositive discriminant (since  $\langle \cdot, \cdot \rangle \geq 0$  and it can have at most one distinct root). Therefore,

$$(2\langle u, v \rangle)^2 - 4 \cdot \langle u, u \rangle \langle v, v \rangle \leq 0 \implies \langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2 \implies \langle u, v \rangle \leq \|u\| \|v\|.$$

We've therefore proven Cauchy-Schwarz inequality for norms over  $\mathbb{R}$ . □

4.4.2 Show that  $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$  for  $x \in \mathbb{R}^n$  and that there are nontrivial examples that attain the equalities.

*Proof.* Clearly  $\|x\|_\infty \leq \|x\|_p$  for any  $p$ , as

$$\|x\|_\infty^p = \max_{1 \leq i \leq n} |x_i|^p \leq \sum_{i=1}^n |x_i|^p = \|x\|_p^p.$$

Instead of showing  $\|x\|_2 \leq \|x\|_1$ , I'd like to show  $\|x\|_q \leq \|x\|_p$  whenever  $p \leq q$ .

(1) If  $\|x\|_p = 1$ , then  $\sum_{i=1}^n |x_i|^p = 1$  which implies each individual  $|x_i|^p \leq 1$ , and so  $|x_i| \leq 1$ . Then,

$$\|x\|_q = \sum_{i=1}^n |x_i|^q \leq \sum_{i=1}^n |x_i|^p = 1.$$

(2) If  $\|x\|_p \neq 1$ , we can first normalize it to  $y := x/\|x\|_p$  so that  $\|y\|_p = 1$ . Then,

$$\|x\|_q = \left\| \underbrace{(\|x\|_p)}_{\in \mathbb{R}_+} y \right\|_q = \|x\|_p \|y\|_q \leq \|x\|_p \text{ by (1).}$$

Therefore  $\|x\|_q \leq \|x\|_p$  for all  $p \leq q$ , and of course  $\|x\|_2 \leq \|x\|_1$  is just one special case. The equalities can be easily obtained if we set  $x$  to be any standard basis for  $\mathbb{R}^n$ , for example  $(1, 0, \dots)$ , in which case all three norms evaluate to 1. □

4.4.3 Show that  $\|x\|_1 \leq n\|x\|_\infty$  and  $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$  for  $x \in \mathbb{R}^n$ .

*Proof.* The first one is immediate since

$$\|x\|_1 = \sum_{i=1}^n |x_i| \leq n \left( \max_{1 \leq i \leq n} |x_i| \right) = n\|x\|_\infty.$$

The second one is analogous. Since both sides are nonnegative it suffices to prove  $\|x\|_2^2 \leq n\|x\|_\infty^2$ . Indeed,

$$\|x\|_2^2 = \sum_{i=1}^n |x_i|^2 \leq n \left( \max_{1 \leq i \leq n} |x_i|^2 \right) = n\|x\|_\infty^2. \quad \square$$

4.4.7 Determine if the following define norms on  $\mathbb{R}^n$ .

- (a)  $\max\{|x_2|, \dots, |x_n|\}$ . **No.**  $(1, 0, \dots) \neq 0$  but this expression evaluates to 0.
- (b)  $\sum_{i=1}^n |x_i|^3$ . **No:** subadditivity is not satisfied. Consider  $(1, \dots, 1)$  which evaluates to  $n$ .  $(2, \dots, 2)$ , however, evaluates to  $8n > n + n$ .
- (c)  $\left(\sum_{i=1}^n \sqrt{|x_i|}\right)^2$ . **No.** Consider  $(0, 1), (1, 1) \in \mathbb{R}^2$ . Then

$$\begin{cases} (4, 0) \mapsto (2+0)^2 = 4 \\ (0, 4) \mapsto (0+2)^2 = 4 \end{cases} \quad \text{but } (4, 0) + (0, 4) = (4, 4) \mapsto (2+2)^2 = 16 > 4 + 4.$$

- (d)  $\max\{|x_1 - x_2|, |x_1 + x_2|, |x_3|, |x_4|, \dots, |x_n|\}$ . **Yes, this is a norm.** Define  $f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  by  $f(x) :=$  the max function described in this problem. Clearly the codomain is  $\mathbb{R}_{\geq 0}$  by construction.
  - (1) Non-degeneracy:  $f(x) \geq 0$  since absolute values are nonnegative. If  $f(x) = 0$  then  $x_3 = \dots = x_n = 0$  and  $|x_1 - x_2| = |x_1 + x_2| = 0$ . This means  $x_1 = x_2 = 0$  as well and thus  $x = 0$ .
  - (2) Absolute homogeneity: this follows directly from the fact that  $|\lambda v| = |\lambda||v|$ .
  - (3) Subadditivity:

$$\begin{aligned} f(x+y) &= \max\{|x_1 + y_1 - x_2 - y_2|, |x_1 + y_1 + x_2 + y_2|, |x_3 + y_3|, \dots, |x_n + y_n|\} \\ &\leq \max\{|x_1 - x_2| + |y_1 - y_2|, |x_1 + x_2| + |y_1 + y_2|, |x_3| + |y_3|, \dots\} \\ &\leq \max\{|x_1 - x_2|, |x_1 + x_2|, |x_3|, \dots\} + \max\{|y_1 - y_2|, |y_1 + y_2|, |y_3|, \dots\} \\ &= f(x) + f(y). \end{aligned}$$

Therefore, weird as it sounds,  $f$  actually defines a norm on  $\mathbb{R}^n$ .

- (e)  $\sum_{i=1}^n 2^{-i}|x_i|$ . **This is yet another norm.** Like above, non-degeneracy and absolute homogeneity are clear. It remains to show subadditivity. Indeed,

$$\begin{aligned} \sum_{i=1}^n 2^{-i}|x_i + y_i| &\leq \sum_{i=1}^n 2^{-i}(|x_i| + |y_i|) \\ &= \sum_{i=1}^n 2^{-i}|x_i| + \sum_{i=1}^n 2^{-i}|y_i| \\ &= \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i|. \end{aligned}$$

4.4.11 Show that the matrix norm  $\|A\|_1$  subordinate to  $\|x\|_1$  ( $x \in \mathbb{R}^n$ ) is

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|.$$

*Proof.* Let  $A$  be given. Let the  $\tilde{j}^{\text{th}}$  column be the one with largest 1-norm. Let  $A_i$  denote the  $i^{\text{th}}$  column

of  $A$ . Then,

$$\begin{aligned} \|A\|_1 &= \sup_{\|x\|_1=1} \|Ax\|_1 = \sup_{\|x\|_1=1} \left\| \sum_{i=1}^n x_i A_i \right\|_1 \\ &\leq \sup_{\|x\|_1=1} \sum_{i=1}^n \|x_i A_i\|_1 = \sup_{\|x\|_1=1} \sum_{i=1}^n |x_i| \|A_i\|_1 \\ &\leq \sup_{\|x\|_1=1} \sum_{i=1}^n |x_i| \|A_{\tilde{j}}\|_1 = \|A_{\tilde{j}}\|_1 \sup_{\|x\|_1=1} \sum_{i=1}^n |x_i| \\ &= \|A_{\tilde{j}}\|_1. \end{aligned}$$

The other direction is much easier: simply consider  $\tilde{x} \in \mathbb{R}^n$  with  $\tilde{x}_i = \delta_{i,\tilde{j}}$ , i.e., 1 for the  $\tilde{j}^{\text{th}}$  entry and 0 otherwise. Then  $\|A\tilde{x}\|_1 = \|A_{\tilde{j}}\|_1$  and the  $\geq$  follows from definition of supremum.  $\square$

- (1) Prove that if  $\|\cdot\|$  is a norm on a vector space, and if we define  $\|x\|' := \alpha\|x\|$  with a fixed  $\alpha > 0$ , then  $\|\cdot\|'$  also defines a norm.

*Proof.* Non-degeneracy is trivial. So is absolute homogeneity since  $\|\lambda x\|' = \alpha\|\lambda x\| = |\lambda|\alpha\|x\| = |\lambda|\|x\|'$ . For triangle inequality,

$$\|x + y\|' = \alpha\|x + y\| \leq \alpha(\|x\| + \|y\|) = \alpha\|x\| + \alpha\|y\| = \|x\|' + \|y\|'. \quad \square$$

- 4.4.24 If the construction in the preceding problem is applied to a subordinate matrix norm, is the resulting norm also a subordinate matrix norm?

**Solution**

Yes.

Nonnegativity is trivial. Now suppose  $\|A\|' = 0$ . Clearly if  $A = 0$  then  $\|A\|' = 0$ . For the converse, if  $\|A\|' = 0$  but  $A$  is not the zero matrix, then some column of  $A$  needs to be nonzero (not all 0's). Consider  $v = (0, \dots, 0, 1/\alpha, 0, \dots)^T$  where the  $1/\alpha$  corresponds to that column. Then  $\|v\| = 1$  and  $Av \neq 0$ , contradiction.

Absolute homogeneity is also clear:  $\|\lambda A\|' = \sup_{\|x\|'=1} \|\lambda Ax\|' = |\lambda| \sup_{\|x\|'=1} \|Ax\|' = |\lambda|\|A\|'$ .

For triangle inequality:

$$\begin{aligned} \|A + B\|' &= \sup_{\|x\|'=1} \|(A + B)x\|' = \sup_{\|x\|'=1} \|Ax + Bx\|' \\ &\leq \sup_{\|x\|'=1} (\|Ax\|' + \|Bx\|') \leq \sup_{\|x\|'=1} \|Ax\|' + \sup_{\|x\|'=1} \|Bx\|' \\ &= \|A\|' + \|B\|'. \end{aligned}$$

Therefore this new norm is still subordinated.  $\square$

- 4.4.29 Prove that if  $A$  has a nontrivial fixed point then  $\|A\| \geq 1$  for any subordinate matrix norm.

*Proof.* Suppose  $Ax = x$  for some  $x \neq 0$ . Normalizing  $x$  to  $\tilde{x} := x/\|x\|$  we see that

$$A\tilde{x} = \tilde{x} \implies \|A\tilde{x}\| = \|\tilde{x}\| = 1 \implies \|A\| = \sup_{\|x\|=1} \|Ax\| \geq \|A\tilde{x}\| = 1. \quad \square$$

4.4.34 For any  $n \times n$  matrix  $A$ , define the **Frobenius norm** to be  $\|A\|_F := \left( \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \right)^{1/2}$ . Show that this defines a norm on the vector space of all  $n \times n$  matrices. What about  $\|A\| := \max_{1 \leq i, j \leq n} |a_{ij}|$ ?

*Proof (Frobenius).* To see that  $\|A\|_F$  defines a norm, simply notice that  $\|A\|_F$  is the same as the 2-norm of a super long vector in  $\mathbb{R}^{n \times n} : (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{nn})^T$ . Therefore  $\|A\|_F$  automatically satisfies all three postulates.

However, the Frobenius norm is not subordinated. For example,  $Ix = x$  for all  $x$ , so if  $\|\cdot\|_F$  were subordinate, one would have  $\|I\|_F = \sup_{\|x\|=1} \|Ix\| = \sup_{\|x\|=1} \|x\| = 1$ , while in reality  $\|I\|_F = \sqrt{n}$  and  $n$  can  $> 1$ .  $\square$

*Proof (max norm).* Non-degeneracy and absolute homogeneity are trivial. For triangle inequality,

$$\max_{1 \leq i, j \leq n} |a_{ij} + b_{ij}| \leq \max_{1 \leq i, j \leq n} (|a_{ij}| + |b_{ij}|) \leq \max_{1 \leq i, j \leq n} |a_{ij}| + \max_{1 \leq i, j \leq n} |b_{ij}|.$$

This norm, unfortunately, is not subordinated, either. For if it were, then

$$\|AB\| = \sup_{\|x\|=1} \|ABx\| \leq \sup_{\|x\|=1} [\|A\| \|Bx\|] \leq \sup_{\|x\|=1} [\|A\| \|B\| \|x\|] = \|A\| \|B\|.$$

However, notice that if we let  $A = B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  then  $AB = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ . Then  $\|A\| = \|B\| = 1$  but  $\|AB\| = 2$ .  $\square$

4.4.37 Prove that for each  $x \in \mathbb{R}^n$ ,

$$\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty.$$

*Remark: the same equation holds for  $x \in \ell^q$  or  $L^q$ ,  $q \geq 1$ .*

*Proof.* On one hand, since each  $|x_i|$  is bounded by the largest one, i.e.,  $\max_{1 \leq i \leq n} |x_i|$ ,

$$\left( \sum_{i=1}^n \|x_i\|^p \right)^{1/p} \leq \left( n \cdot \max_{1 \leq i \leq n} |x_i|^p \right)^{1/p} = n^{1/p} \max_{1 \leq i \leq n} |x_i| = n^{1/p} \|x\|_\infty.$$

On the other hand, since  $\sum_{i=1}^n$  includes the largest one and other  $|x_i|$ 's are also nonnegative,

$$\left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \geq \left( \max_{1 \leq i \leq n} |x_i|^p \right)^{1/p} = \max_{1 \leq i \leq n} |x_i| = \|x\|_\infty.$$

Since  $p^{1/p} = \exp((1/p) \log(p))$  and L'Hôpital's rule gives

$$\lim_{p \rightarrow \infty} \frac{\log(p)}{p} = \lim_{p \rightarrow \infty} \frac{1/p}{1} = 0,$$

we know  $\lim_{p \rightarrow \infty} p^{1/p} = e^0 = 1$ . Therefore, by squeeze theorem,  $\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty$ .  $\square$

