MATH 501 Homework 8

Qilin Ye, March 26, 2021

Pseudocode Implementation

```matlab
M = input("Number of iterations: ");
A = [2,-1,0;1,6,-2;4,-3,8];
b = [2 -4 5];
x = [0; 0; 0];
u = [0; 0; 0];
n = size(x,1);
Iteration = zeros(M,1);
Jacobi_Method_Approximation = zeros(M,n);

for k = 1:M
    for i = 1:n
        d = 1 / A(i,i);
        b(i) = d * b(i);
        for j = 1:n
            A(i,j) = d * A(i,j);
        end
        u(i) = b(i) - A(i,1:i-1) * x(1:i-1) - A(i,i+1:n) *
            x(i+1:n);
    end
    for i = 1:n
        x(i) = u(i);
    end
    Iteration(k) = k;
    Jacobi_Method_Approximation(k,1:n) = x';
end

disp(' ');
disp(table(Iteration, Jacobi_Method_Approximation));
```

Command Window

Number of iterations: 13

Iteration	Jacobi_Method_Approximation		
1	1	-0.66667	0.625
2	0.66667	-0.625	-0.125
3	0.6875	-0.81944	0.057292
4	0.59028	-0.76215	-0.026042
5	0.61892	-0.77373	0.044054
6	0.61314	-0.75514	0.025391
7	0.62243	-0.76039	0.035256
8	0.6198	-0.75865	0.028637
9	0.62067	-0.76042	0.030603
10	0.61979	-0.75991	0.029505
11	0.62004	-0.76013	0.030139
12	0.61994	-0.75996	0.029929
13	0.62002	-0.76001	0.030047

```matlab
%Gauss-Seidel
M = input("Number of iterations: ");
A = [2,-1,0;1,6,-2;4,-3,8];
b = [2 -4 5];
x = [0; 0; 0];
n = size(x,1);

Iteration = zeros(M,1);
Gauss_Siedel_Approximation = zeros(M,n);

for k = 1:M
    for i = 1:n
        x(i) = (b(i) - A(i,1:i-1) * x(1:i-1) - A(i,i+1:n)
            * x(i+1:n)) / A(i,i);
    end
    Iteration(k) = k;
    Gauss_Siedel_Approximation(k,1:n) = x';
end
disp(' ');
disp(table(Iteration, Gauss_Siedel_Approximation));
```

Command Window

Number of iterations: 13

Iteration	Gauss_Siedel_Approximation		
1	1	-0.83333	-0.1875
2	0.58333	-0.82639	0.023438
3	0.58681	-0.75666	0.047852
4	0.62167	-0.75433	0.031291
5	0.62284	-0.76004	0.028566
6	0.61998	-0.76047	0.029833
7	0.61976	-0.76002	0.030113
8	0.61999	-0.75996	0.030019
9	0.62002	-0.76	0.029991
10	0.62	-0.76	0.029998
11	0.62	-0.76	0.030001
12	0.62	-0.76	0.03
13	0.62	-0.76	0.03

Textbook Problems

4.4.44 Let A be an $m \times n$ matrix. We interpret A as a linear map from $\mathbb{R}^n$ with $\|\cdot\|_1$ to $\mathbb{R}^m$ with $\|\cdot\|_\infty$. What is $\|A\|$ under these circumstances?

Solution

Claim: $\|A\|$ defined this way is simply $\max\{|a_{i,j}| : a_{i,j} \in A\}$. Indeed, $\|Ax\|_\infty$ only cares about the entry that has the largest absolute value. Let it be the k^{th} component of Ax, say. Let $x \in \mathbb{R}^n$ be any vector with $\|x\|_1 = 1$. By definition, we want to find the supremum of the absolute value of

$$(Ax)_k = \begin{bmatrix} a_{k,1} & \cdots & a_{k,n} \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T,$$

where

$$\|x\|_1 = 1 \implies \sum_{i=1}^n |x_i| = 1.$$

If we assume $|a_{k,\ell}| > |a_{k,\ell'}|$ for all $\ell' \neq \ell$, it immediately follows that

$$-|a_{k,\ell}| \leqslant \sum_{i=1}^n a_{k,i} x_i \leqslant |a_{k,\ell}|.$$

Furthermore, one of the inequalities is always obtained by setting $x_i := \delta_{i,\ell}$. Therefore $\|A\|$ is indeed given by the biggest possible $|a_{i,j}|$. □

4.4.47 Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let A be an $n \times n$ matrix. Put $\|x\|' := \|Ax\|$. What are the precise conditions on A to ensure that $\|\cdot\|'$ is also a norm?

Solution

Claim: $\|\cdot\|'$ is a norm if and only if A is invertible.
For $\implies$, if $\|\cdot\|'$ is a norm, then it is non-degenerate. Hence if $x \neq 0$ then $\|x\|' = \|Ax\| \neq 0$. By the non-degeneracy of $\|\cdot\|$ we know $Ax \neq 0$, and thus A needs to be invertible.
For $\impliedby$, assume A is invertible. By above, we see $\|\cdot\|'$ is indeed non-degenerate as $x \neq 0 \implies \|Ax\| = \|x\|' \neq 0$. Absolute homogeneity follows directly from that of $\|\cdot\|$:

$$\|\lambda x\|' = \|\lambda Ax\| = |\lambda|\|Ax\| = |\lambda|\|x\|'$$

and triangle inequality as well:

$$\|x+y\|' = \|A(x+y)\| = \|Ax + Ay\| \leqslant \|Ax\| + \|Ay\| = \|x\|' + \|y\|'.$$ □

4.4.52 Prove that if A is nonsingular then there exists $\delta > 0$ with the property that $A + E$ is nonsingular for all matrices E satisfying $\|E\| < \delta$.

Proof. First notice that the determinant is a continuous function from $\mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ (or, equivalently, from $\mathbb{R}^{n\times n} \to \mathbb{R}$):

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(i),i}.$$

(Indeed, we can append the row vectors of A to obtain $(a_{1,1},\dots,a_{1,n},a_{2,1},\dots,a_{2,n},\dots,a_{n,1},a_{n,n}) \in \mathbb{R}^{n\times n}$.) For notational clarity, given $\hat{a} \in \mathbb{R}^{n\times n}$ as defined above, we denote $\hat{a}_{i,j}$ by the (i,j) entry of A. Suppose A is nonsingular, i.e., $\det(A) \neq 0$. By the continuity of $\det(\cdot)$, there exists an open neighborhood U of $\hat{a}$ such that $|\det(B) - \det(A)| < |\det(A)|/2$ for all $B \in U$. Therefore all $B \in U$ are also invertible! It remains to show that we can find a δ. Indeed, we can define $C_{n\times n}$ by

$$c_{i,j} = \inf_{\hat{b}\notin U} \frac{|\hat{a}_{i,j} - \hat{b}_{i,j}|}{2}.$$

It follows immediately that $C \in U$ and so is any matrix C' that is entry-wise absolutely bounded by C, i.e., if $|c'_{i,j}| \leqslant c_{i,j}$ for all i,j. Therefore, if we define $\delta := \|C\|$, the claim follows. □

4.4.55 Prove that if A is nonsingular, then there is a singular matrix with distance $\|A^{-1}\|^{-1}$ of A.

4.5.1 Prove that the set of invertible $n\times n$ matrices is an open set in the set of all $n \times n$ matrices. Thus, if A is invertible, then there is a positive ϵ such that every matrix B satisfying $\|A - B\| < \epsilon$ is also invertible.

Proof. This has been shown in Exercise 4.4.52. □

4.5.2 Prove that if A is invertible and $\|B - A\| < \|A^{-1}\|^{-1}$ then B is invertible.

Proof. By assumption, $\|B-A\|\|A^{-1}\| < 1$, and so by Theorem 4.5.1, $I-(B-A)(A^{-1}) = -BA^{-1}$ is invertible. Then it follows that B must be invertible. □

4.5.8 Prove that if $\|A\| < 1$ then

$$(I + A)^{-1} = I - A + A^2 - A^3 + \dots.$$

Proof. This directly follows from Theorem 4.5.1 by noticing $\|-A\| = \|A\| < 1$ and that

$$(-A)^k = (-1)^k A^k.$$ □

4.5.14 Prove that if $\inf\limits_{\lambda\in\mathbb{R}} \|I - \lambda A\| < 1$ then $\|A\|$ is invertible.

Proof. By assumption, there exists some $\lambda_1 \in \mathbb{R}$ such that $\|I - \lambda_1 A\| < 1$. Notice that

$$I - \lambda_1 A = I - (\lambda_1 I)A.$$

Theorem 4.5.2 gives the invertibility of both $\lambda_1 I$ and A (so we are done). □

4.5.20 Show that the sequence of functions $x_n(t) = t^n$ on $[0,1]$ has properties $\|x_n\|_\infty = 1$ and $\|x_n\|_1 \to 0$ as $n \to 0$.

Proof. The L^∞ is clear as $\|x_n\|_\infty = |x(1)| = 1$ for all n. On the other hand,

$$\|x_n\|_1 = \int_0^1 |t^n| \,\mathrm{d}t = \int_0^1 t^n \,\mathrm{d}t = \frac{1}{n+1} \to 0 \text{ as } n \to \infty.$$

This shows that convergence in L^∞ does not imply that in L^1. □

4.5.21 Prove that if $\|AB - I\| < 1$ then $2B - BAB$ is a better approximation of A^{-1} than B in the sense that $A(2B - BAB)$ is closer to I.

Proof. Recall from Theorem 4.5.1 that

$$(AB)^{-1} = \sum_{k=0}^{\infty} (I - AB)^k \implies I = AB \sum_{k=0}^{\infty} (I - AB)^k.$$

It follows that

$$I - A(2B - BAB) = I - AB - AB(I - AB) = AB \sum_{k=2}^{\infty} (I - AB)^k.$$

By the submultiplicativity of $\|\cdot\|$, we have

$$\|I - A(2B - BAB)\| = \left\|AB \sum_{k=2}^{\infty} (I - AB)^k\right\| \leqslant \|I - AB\| \left\|AB \sum_{k=1}^{\infty} (I - AB)^k\right\|$$

where the last $\|\cdot\|$ on the RHS is nothing but $\|I - AB\|$. Since $\|I - AB\| \leqslant 1$ we conclude that

$$\|I - A(AB - BAB)\| \leqslant \|I - AB\|^2 < \|I - AB\|.$$ □

4.5.31 Prove that if p is a polynomial without constant term such that

$$\|I - p(A)\| < 1$$

then A is invertible.

Proof. Obvious. We can write $p(A) := A \cdot q(A)$ thanks to the absence of constant term. Then Theorem 4.5.2 gives the claim. □