MATH 501 Homework 9

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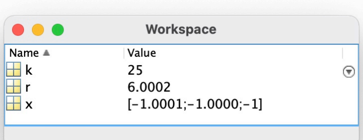
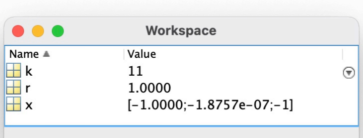
Pseudocode Implementation

```
% Power method, needs phi();

A = [6,5,-5;2,6,-2;2,5,-1];
x = [-1,1,1]';
n = size(A);
n = n(1);

for k = 1:25
    y = A*x;
    r = phi(y) / phi(x);
    x = y / abs(max(y));
end
clearvars -except r x k

% --------------------------------

% Inverse power method

A = [6,5,-5;2,6,-2;2,5,-1];
x = [3,7,-13]';
n = size(A);
n = n(1);

[L, U] = LU_Decomp(A);

for k = 1:11
    y = Iterate(U,L,x);
    x = y / max(abs(y));
    r = phi(y) / phi(x);
end
clearvars -except r x k
```






```
function y = phi(x)
  y = x(2);
end
%Directly from HW5
function [L, U] = LU_Decomp(A)
    n = size(A);
    n = n(1);
    L = zeros(n);
    U = zeros(n);
    for k = 1:n
        L(k,k) = 1;
        for j = k:n
            A_kj_Old = A(k,j);
            for s = 1:k-1
                A_kj_Old = A_kj_Old - L(k,s) * U(s,j);
            end
            U(k,j) = A_kj_Old;
        end

        for i = k+1:n
            A_ik_Old = A(i,k);
            for s = 1:k-1
                A_ik_Old = A_ik_Old - L(i,s) * U(s,k);
            end
            L(i,k) = A_ik_Old / U(k,k);
        end
    end
end

function x = Iterate(U,L,b)
    n = size(U);
    n = n(1);
    z = zeros(n);
    for i=1:n
       b_i_Old = b(i);
       for j = 1:i-1
           b_i_Old = b_i_Old - L(i,j) * z(j);
       end
       z(i) = b_i_Old;
    end
    x = zeros(n,1);
    for i = 0:n-1
        z_i_Old = z(n-i);
        for j = n+1-i:n
            z_i_Old = z_i_Old - U(n-i,j) * x(j);
        end
        x(n-i) = z_i_Old / U(n-i,n-i);
    end
end
```

Textbook Problems

4.6.1 Prove that if A is diagonally dominant and if Q is chosen as in the Jacobi method, then

$$\rho(I - Q^{-1}A) < 1.$$

Proof. If Q simply consists of the diagonal entries of A then Q^{-1} acts on $a_{i,j}$ by $a_{i.j} \mapsto a_{i,j}/a_{i,i}$. It follows that the diagonal entries of $Q^{-1}A$ are all 1 and, since the ratios between entries in the same row remain unchanged, by A's diagonal dominance

$$(Q^{-1}A)_{i,i} = 1 = \frac{|a_{i,i}|}{|a_{i,i}|} > \Big(\sum_{j\neq i}|a_{i,j}|\Big)/|a_{i,i}| = \sum_{j\neq i}\frac{|a_{i,j}|}{|a_{i,i}|} = \sum_{j\neq i}|(Q^{-1}A)_{i,j}|$$

and so $Q^{-1}A$ is also diagonal dominant. It follows that $\|I - Q^{-1}A\|_\infty < 1$, and $\rho(I - Q^{-1}A) \leqslant \|I - Q^{-1}A\|_\infty$ by definition of infimum. □

4.6.5 Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Let S be an $n \times n$ nonsingular matrix. Define $\|x\|' = \|Sx\|$. Prove that $\|\cdot\|'$ is a norm.

Proof. Non-degeneracy is clear as $x \neq 0 \Rightarrow Sx \neq 0 \Rightarrow \|Sx\| > 0$ and $\|Sx\| = 0 \iff Sx = 0 \iff x = 0$.
Absolute homogeneity:

$$\|\lambda x\|' = \|S(\lambda x)\| = \|\lambda(Sx)\| = |\lambda|\|Sx\| = |\lambda|\|x\|'.$$

For triangle inequality:

$$\|x + y\|' = \|S(x + y)\| = \|Sx + Sy\| \leqslant \|Sx\| + \|Sy\| = \|x\|' + \|y\|'.$$ □

4.6.6 Let $\|\cdot\|$ be a subordinate matrix norm and let S be a nonsingular matrix. Define $\|A\|' = \|SAS^{-1}\|$. Show that $\|\cdot\|'$ is a subordinate matrix norm.

Proof. $\|A\|' = \|SAS^{-1}\| = \sup_{\|x\|=1}\|SAS^{-1}x\| = \sup_{\|Sx\|=1}\|SAx\| = \sup_{\|x\|'=1}\|Ax\|'$, where the second step takes advantage of the fact $SS^{-1}x = x$ and the third uses the result from 4.6.5. □

4.6.10 Which of the norm axioms are satisfied by the spectral radius function ρ and which are not?

Solution

Non-degeneracy is not satisfied: the eigenvalues of the nonzero matrix $\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ are both 0 so $\rho(\cdot) = 0$.

Absolute homogeneity is not satisfied: suppose λ_i is the maximal eigenvalue of A with $Ax_i = \lambda_i x$. We then have

$$(-A)x_i = -(Ax_i) = -\lambda_i x_i \neq |-1| \cdot \lambda_i x_i.$$

Triangle inequality is not satisfied, either. Consider $A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$. Clearly $\rho(A) = \rho(B) = 0$ but $A + B = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ which has $\rho(\cdot) = 1$.

4.6.16 Prove that if A is nonsingular then AA^* is positive definite.

Proof. A^* is also nonsingular. Thus, for $x \neq 0$, $A^*x \neq 0$ and $x^*AA^*x = (A^*x)^*(A^*x) = \sqrt{\|A^*x\|} > 0$. □

4.6.18 Prove that if A is positive definite, then so are $A^2, A^3, \ldots$ as well as $A^{-1}, A^{-2}, \ldots$.

Proof. If A is positive definite then its eigenvalues are strictly positive. Notice that if $Ax = \lambda x$ then $A^k x = A^{k-1}(Ax) = A^{k-1}\lambda x = \cdots = \lambda^k x$. Thus all eigenvalues of A^k are also positive, and this implies the positive definiteness of A^k: for an eigenvector x corresponding to eigenvalue $\lambda > 0$ of B, we have

$$x^* Bx = x^*(\lambda x) = \lambda(x^* x) > 0,$$

and the result for more general x's (i.e., linear combination of eigenvectors) follow from linearity of matrix operation. For powers of A^{-1}, simply notice that the eigenvalues of A^{-1} are $1/\lambda_i$'s. □

5.1.1 Let A be an $n \times n$ matrix with P's column being its linearly independent eigenvectors. What is $P^{-1}AP$?

Solution

$P^{-1}AP = P^{-1}(P\Lambda P^{-1})P = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

5.1.3 In the power method, let $r_k = \varphi(x^{(k+1)})/\varphi(x^{(k)})$. We know that $\lim_{k\to\infty} r_k = \lambda_1$. Show that the relative errors obey

$$\frac{r_k - \lambda_1}{\lambda_1} = \left(\frac{\lambda_2}{\lambda_1}\right)^k c_k$$

where $\{c_k\}$ is convergent.

Proof. Since

$$r_k = \frac{\varphi(x^{(k+1)})}{\varphi(x^{(k)})} = \frac{\lambda_1^{k+1}(\varphi(u^{(1)}) + \varphi(\epsilon^{(k+1)}))}{\lambda_1^k(\varphi(u^{(1)}) + \varphi(\epsilon^{(k)}))} = \lambda_1 \frac{\varphi(u^{(1)}) + \varphi(\epsilon^{(k+1)})}{\varphi(u^{(1)}) + \varphi(\epsilon^{(k)})},$$

we have

$$\begin{aligned}
\frac{r_k - \lambda_1}{\lambda_1} &= \frac{\varphi(u^{(1)}) + \varphi(\epsilon^{(k+1)})}{\varphi(u^{(1)}) + \varphi(\epsilon^{(k)})} - 1 = \frac{\varphi(\epsilon^{(k+1)}) - \varphi(\epsilon^{(k)})}{\varphi(u^{(1)}) + \varphi(\epsilon^{(k)})} \\
&= \frac{\varphi[(\lambda_2/\lambda_1)^{k+1}u^{(2)} - (\lambda_2/\lambda_1)^k u^{(2)} + \cdots + (\lambda_n/\lambda_1)^{k+1}u^{(n)} - (\lambda_n/\lambda_1)^k u^{(n)}]}{\varphi(u^{(1)} + \epsilon^{(k)})} \\
&= \frac{\varphi[(\lambda_2/\lambda_1)^k(\lambda_2/\lambda_1 - 1)u^{(2)} + \ldots]}{\varphi(u^{(1)} + \epsilon^{(k)})} \\
&= \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{\varphi[(\lambda_2/\lambda_1 - 1)u^{(2)} + (\lambda_3/\lambda_2)^k(\lambda_3/\lambda_1 - 1)u^{(3)} + \ldots]}{\varphi(u^{(1)} + \epsilon^{(k)})} \\
&= \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{\varphi(\lambda_2/\lambda_1 - 1)u^{(2)}}{\varphi(u^{(1)})}.
\end{aligned}$$

□

5.1.4 Show that $r_{k+1} - \lambda = (c + \delta_k)(r_k - \lambda_1)$ where $|c| < 1$ and $\lim_{n\to\infty} \delta_k = 0$ so that the Aitken acceleration is applicable.

Proof. First we compute $r_{k+1} - \lambda_1$:

$$\begin{aligned}
r_{k+1} - \lambda_1 &= \frac{\varphi(x^{(k+2)}) - \lambda_1\varphi(x^{(k+1)})}{\varphi(x^{(k+1)})} = \frac{\varphi(x^{(k+2)} - \lambda_1 x^{(k+1)})}{\varphi(x^{(k+1)})} \\
&= \frac{\varphi\left[\lambda_1^{k+1}(a_1u^{(1)} + \epsilon^{(k+2)}) - \lambda_1^{k+2}(a_1u^{(1)} + \epsilon^{(k+1)})\right]}{\varphi(\lambda_1^{k+1}(a_1u^{(1)} + \epsilon^{(k+1)}))} \\
&= \frac{\lambda_1^{k+2}\varphi(\epsilon^{(k+2)} - \epsilon^{(k-1)})}{\lambda_1^{k+1}\varphi(a_1u^{(1)} + \epsilon^{(k+1)})} \\
&= \lambda_1 \frac{\varphi(\epsilon^{(k+2)} - \epsilon^{(k+1)})}{\varphi(a_1u^{(1)} + \epsilon^{(k+1)})}.
\end{aligned}$$

Therefore

$$\frac{r_{k+1} - \lambda_1}{r_k - \lambda_1} = \frac{\varphi(a_1u^{(1)} + \epsilon^{(k)})}{\varphi(\epsilon^{(k+1)} - \epsilon^{(k)})} \cdot \frac{\varphi(\epsilon^{(k+2)} - \epsilon^{(k-1)})}{\varphi(a_1u^{(1)} + \epsilon^{(k+1)})}$$

which indeed converges to a constant as a constant with absolute value < 1 as $k \to \infty$. □

5.1.9 What can you prove about Aitken acceleration if the sequence $\{r_n\}$ satisfies only the hypothesis $|r_{n+1} - r| \leqslant c|r_n - r|$ with $0 < c < 0.2$?

Solution

The faster convergence by Aitken acceleration still holds as certainly any sequence bounded by $(-0.2, 0.2)$ can be written as $a_n = c' + \delta_n$ for some fixed c' and $\delta_n \to 0$.