Contents

7.1 Introduction – Why *Uniform* Convergence of Functions?

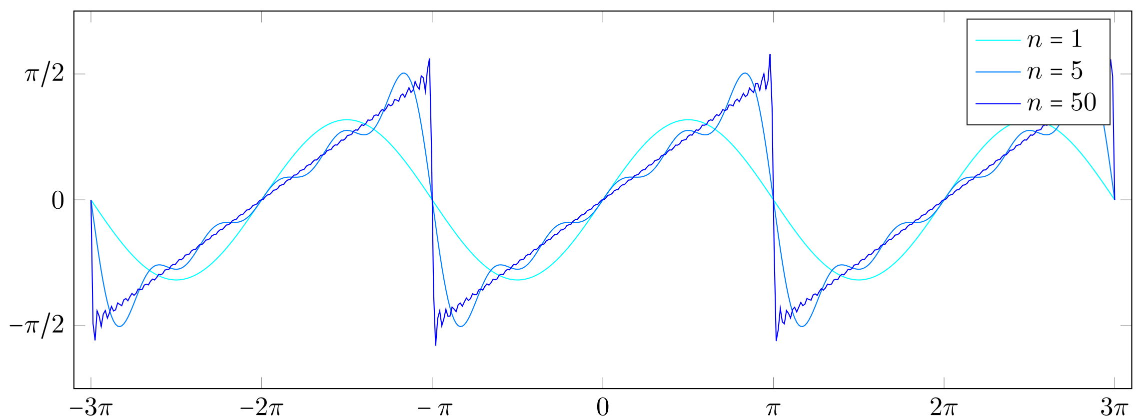
A short history[1] of why we need a stronger mode of convergence of functions called *uniform convergence* and in particular why pointwise convergence of functions isn't sufficient:

(1) In 1821, Cauchy hypothesized that if $f_n : \mathbb{R} \to \mathbb{R}$ are continuous and $\sum_{n=1}^{\infty} f_n$ converges to f pointwise, i.e., if $\sum_{n=1}^{\infty} f_n(x) \to f(x)$ for all $x \in \mathbb{R}$, then f is continuous.

(2) In 1826, Abel provided a counterexample to above: consider the *Fourier series* of the *discontinuous* function $f(x) = x/2$ on $(-\pi, \pi)$ but extended 2π-periodically to $\mathbb{R}$ with $f(2\pi) = 0$. (The graph consists of parallel line segments. See graph below.)

It was known that

$$f(x) = \sin(x) = \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(nx).$$



From this we see that the pointwise limit of a sequence of continuous functions may be discontinuous. We would also run into problems if we try to differentiate the series term by term, obtaining

$$\cos(x) - \cos(2x) + \cos(3x) - \ldots$$

which diverges for most x, whereas the derivative of f exists *almost everywhere* and equals to $1/2$.
We will now formally introduce the notion of uniform convergence and derive many of its nice properties — for example, how uniform convergence interacts with limits, series, integrals, and derivatives in ways that (standard) convergence may fail to.

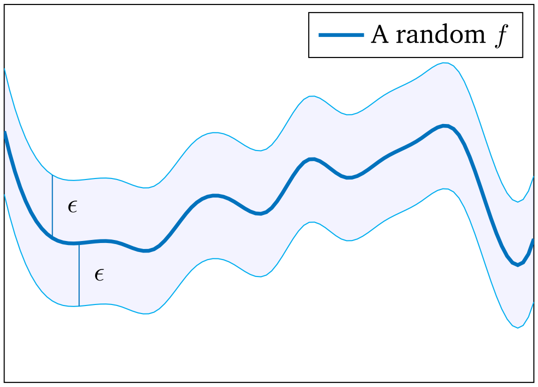
7.2 Uniform Convergence of Functions

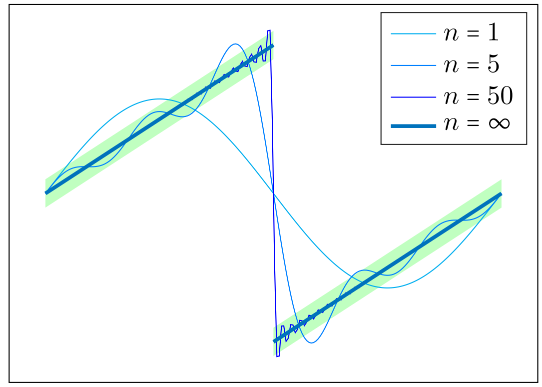
Definition 12.10: Pointwise and Uniform Convergence of Functions

Let $(f_n)_{n\geqslant 1}$ be a sequence of functions. We say that $f_n : X \to Y$ **converges** to $f : X \to Y$ **pointwise** if $f_n(x) \to f(x)$ for each $x \in X$. We say f_n **converges** to f **uniformly** (on X)[2] if $\sup_{x\in X} |f_n(x) - f(x)| \to 0$.
Equivalently, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$ and all $n \geqslant N$. (*In particular, uniform convergence implies pointwise convergence.*)

[1] From my 425b, taught by Prof. Andrew Manion. Lecture notes, sample HWs, and exams can be found on my website.

To illustrate the difference between convergence and uniform convergence: if $f_n \to f$ uniformly, then for all large enough n's, the corresponding f_n's need to be contained by the "ϵ-tube[3]" of f, as shown in the left figure. Letting $\epsilon \to 0$, it becomes clear that all $x \in X$ need to synchronously approach their corresponding limits on f, hence the word "uniform". The counterexample provided by Abel clearly fails to satisfy this criterion: near π, every single f_n jumps out of the green ϵ-tube, so the convergence is not uniform, which is (we'll show soon) precisely why Cauchy's hypothesis is false.



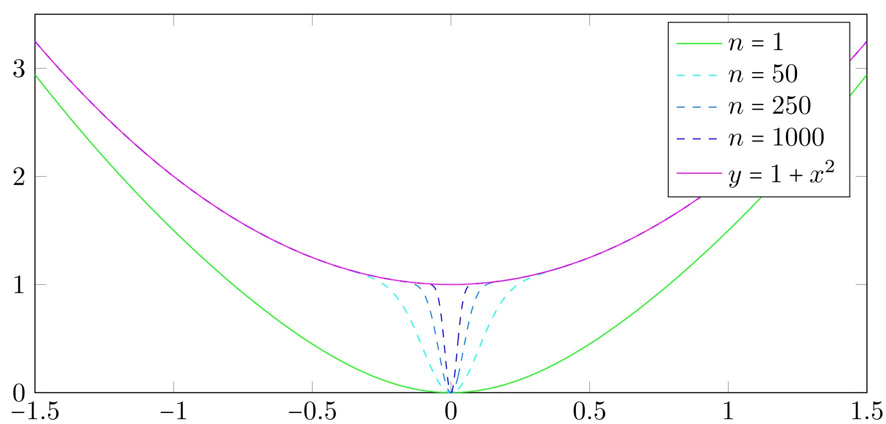


Example 12.11: Examples Where Convergence isn't Good Enough.

(1) (Another example that) continuity may fail if only assuming pointwise convergence: define

$$f_n(x) := x^2 \sum_{k=0}^{n} (1 + x^2)^{-k} \qquad \text{and} \qquad f(x) := \begin{cases} \dfrac{x^2}{1 - 1/(1 + x^2)} = 1 + x^2 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(Note that $(1 + x^2) > 1$ as $x \neq 0$, and since $-k < 0$, the sum forms a geometric series, so indeed $f_n(x) \to f(x)$ pointwise.) However, it is clear that although each f_n is continuous (they are in fact C^∞, i.e., smooth!), f is not at 0 (not even C^1).



(2) Limit of integral need not equal integral of limit if only assuming pointwise convergence: if

$$f_n(x) := n^2 x (1 - x^2)^n \qquad \text{on } [0, 1],$$

[2]Notations include $f_n \rightrightarrows f$, but I will instead say "$f_n \to f$ uniformly" every time to strengthen memory.

[3]I first saw this in Pugh's book, and I loved this description. Vivid, intuitive, and self-explanatory.

then $f_n(x) \to 0$ for all $x \in [0,1]$ (see Rudin, Theorem 3.20(d)). However,

$$\int_0^1 f_n(x)\,\mathrm{d}x = n^2 \int_0^1 (\underbrace{1-x^2}_{=:y})^n\,\mathrm{d}x = \frac{n^2}{2}\int_0^1 y^n\,\mathrm{d}y = \frac{n^2}{2n+1} \to \infty,$$

so this example demonstrates

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = 0 \neq \lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x.$$

Theorem 12.12: Uniformly Convergent & Cauchy w.r.t. Sup Metric

For the space of real-valued functions on X, we define a metric, called the **sup metric**, written $\|\cdot\|_{\sup}$, by

$$\|f\|_{\sup} := \sup_{x\in X}|f(x)|.$$

Then (*think of completeness, i.e., convergence* $\Leftrightarrow$ *Cauchy-ness but w.r.t. function norms; we will talk about this in detail in Theorem 13.3.*)

$$f_n \to f \text{ uniformly} \iff (f_n)_{n\geqslant 1} \text{ forms a Cauchy sequence w.r.t. } \|\cdot\|_{\sup}.$$

In other words, the RHS states that, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\| = \sup_{x\in X}|f_n(x) - f_m(x)| < \epsilon$ for all $m, n \geqslant N$.

Future reference: Uniform convergence & derivatives, Weierstraß M-Test, Arzelá-Ascoli Theorem and its proof

Proof. We first show $\Longleftarrow$ using an $\epsilon/2$ argument. Given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|f_n - f\| < \epsilon/2$ for all $n \geqslant N$. Therefore, for $m, n \geqslant N$, and any x, we have

$$|f_n(x) - f_m(x)| \leqslant \underbrace{|f_n(x) - f(x)|}_{<\epsilon/2} + \underbrace{|f(x) - f_m(x)|}_{<\epsilon/2} < \epsilon.$$

Therefore, taking supremum over all x, we obtain

$$\|f_n - f_m\|_{\sup} = \sup_{x\in X}|f_n(x) - f_m(x)| \leqslant \epsilon.$$

For $\Longleftarrow$, also let $\epsilon > 0$ be given. By "Cauchy-ness"[4], there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon \qquad \text{for all } x \in X \text{ and all } m, n \geqslant N. \tag{Δ}$$

In particular, this means that, for all x, the sequence of *real numbers* $(f_n(x))_{n\geqslant 1}$ is Cauchy. Since $\mathbb{R}$ is complete, there exists some real number, which we call $f(x)$, such that $f_n(x) \to f(x)$. *Therefore we can define a function f by setting its value at x to be $f(x)$, the limit of $(f_n(x))_{n\geqslant 1}$.*

Taking $\lim_{m\to\infty}$ in (Δ), we see that $f_m(x) \to f$, and since $|\cdot|$ is continuous, $|f_n(x) - f_m(x)| \to |f_n(x) - f(x)|$. By Theorem 5.8, we obtain

$$|f_n(x) - f(x)| \leqslant \epsilon \qquad \text{for all } x \in X \text{ and } n \geqslant N.$$

which proves the uniform convergence. □

[4] I quoted this "Cauchy-ness" because we haven't taken a rigorous approach to prove the important properties of $\|\cdot\|_{\sup}$.

8.6 Weierstraß' Monster

Having just studied the *Devil*, we now conclude the course with a *monster*[!]

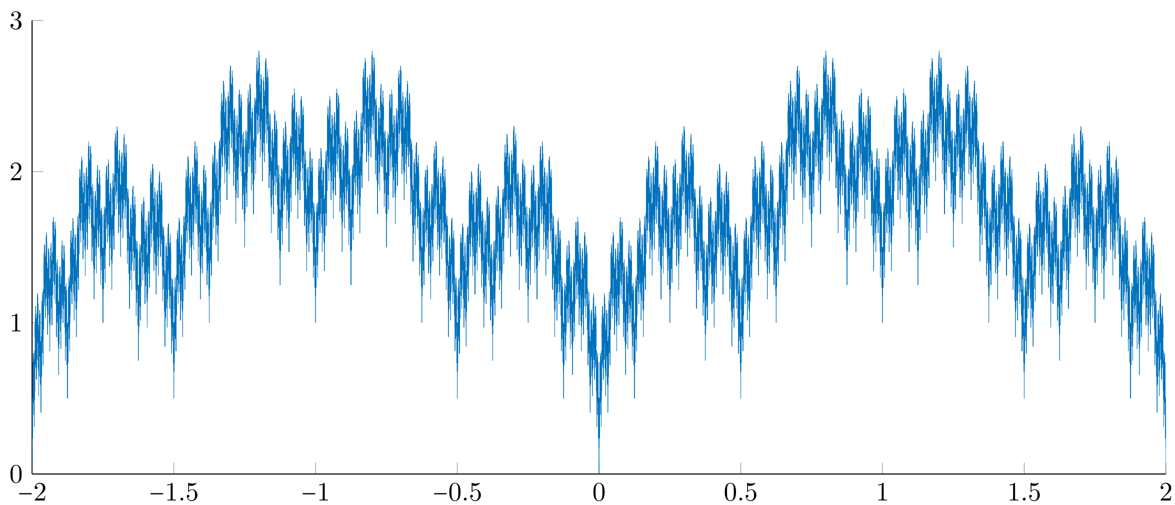
Theorem 15.10: Weierstraß' Monster

There exists $f \in C(\mathbb{R})$ that is *nowhere*[!] differentiable.

When Weierstraß first published this result in 1872, his contemporaries denounced such construction and considered it a **monster**. Henri Poincaré griped that it was "an outrage against common sense," and Charles Hermite famously wrote:

> *I turn with terror and horror from this lamentable scourage of continuous functions with no derivatives.*

While the original *Weierstraß' monster* involves an infinite series of transcendental functions (cosines), we will investigate a purely algebraic one, as seen in both Rudin's Theorem 7.18 and Pugh's Theorem 4.31.
The main idea is simple — we will add up some **sawtooth functions** to make the partial sum non-differentiable at more and more points, eventually resulting in all of $\mathbb{R}$.



Zoom in to see its craziness! This graph consists of $4^7 + 1 = 16385$ data points. This (and almost all other diagrams in this notes) is a vector graph and zooming in will not blur the image, only to make it clearer.

Construction of the *Monster*

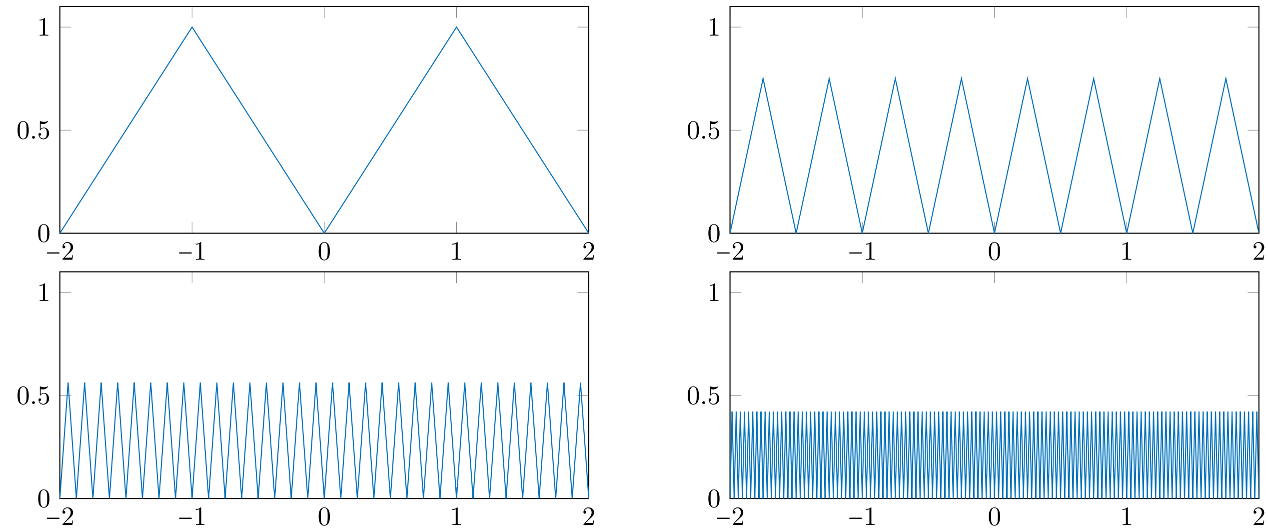
We begin with a **sawtooth function** $\varphi_0(x)$ defined by

$$\varphi_0(x) := |x - 2n| \text{ for } x \in [2n-1, 2n+1) = \begin{cases} x - 2n & x \in [2n, 2n+1) \\ 2n+2-x & x \in [2n+1, 2n+2) \end{cases}$$

The graphs clearly illustrates why it is called the "sawtooth function". Next, we define $\varphi_n(x)$ iteratively by

$$\varphi_n(x) = \left(\frac{3}{4}\right)^n \varphi_0(4^n x).$$

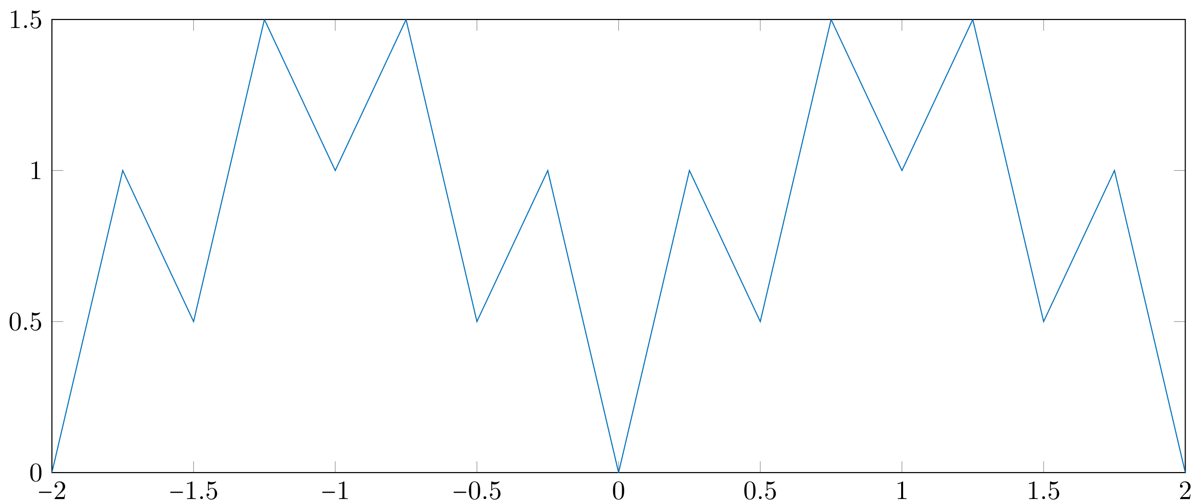
Below are the graphs for $\varphi_0, \varphi_1, \varphi_2$, and φ_3 on $[-2, 2]$. The exponential growth is just scary...

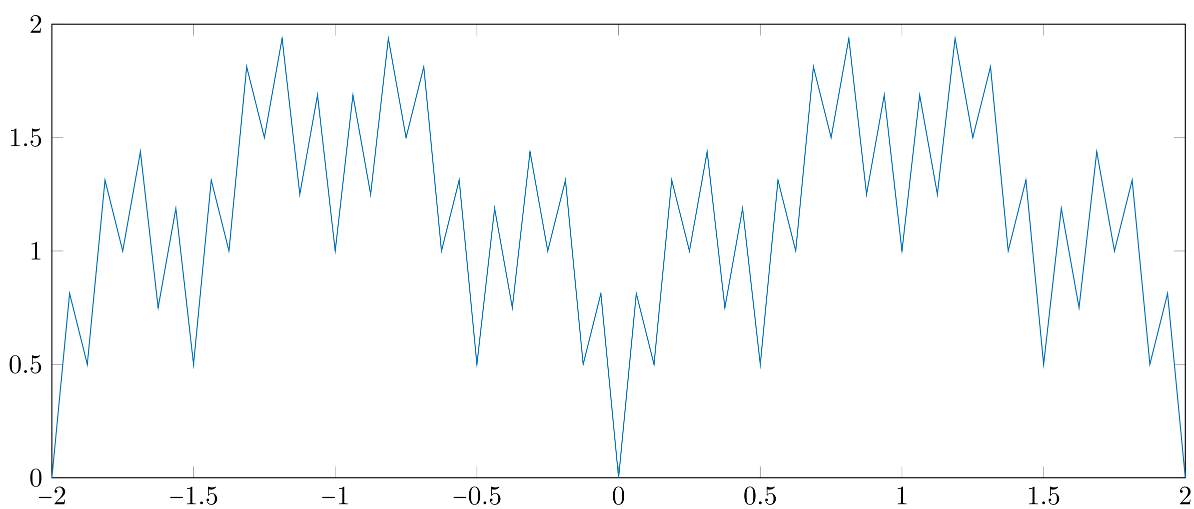


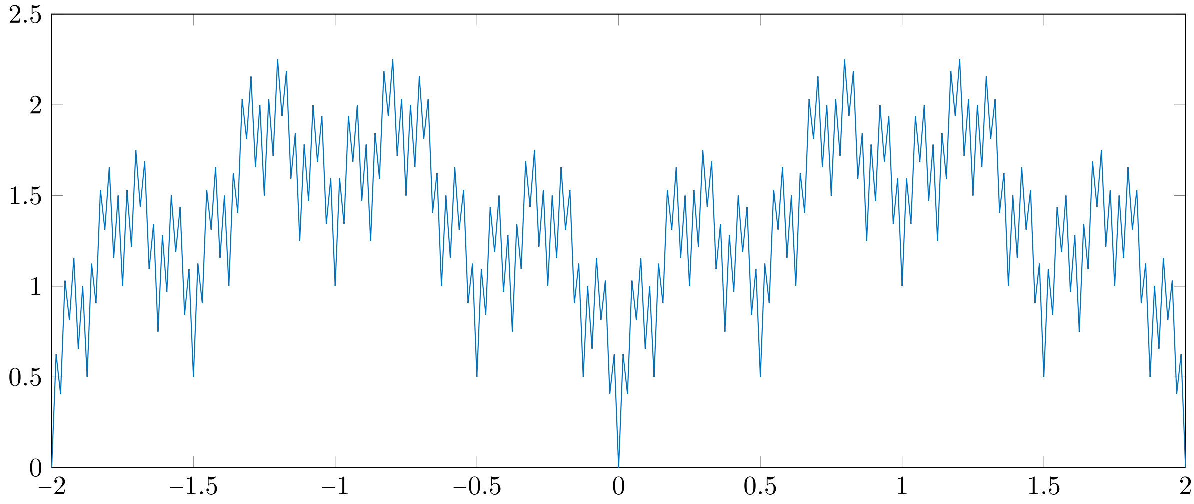
We define the *monster* function to be

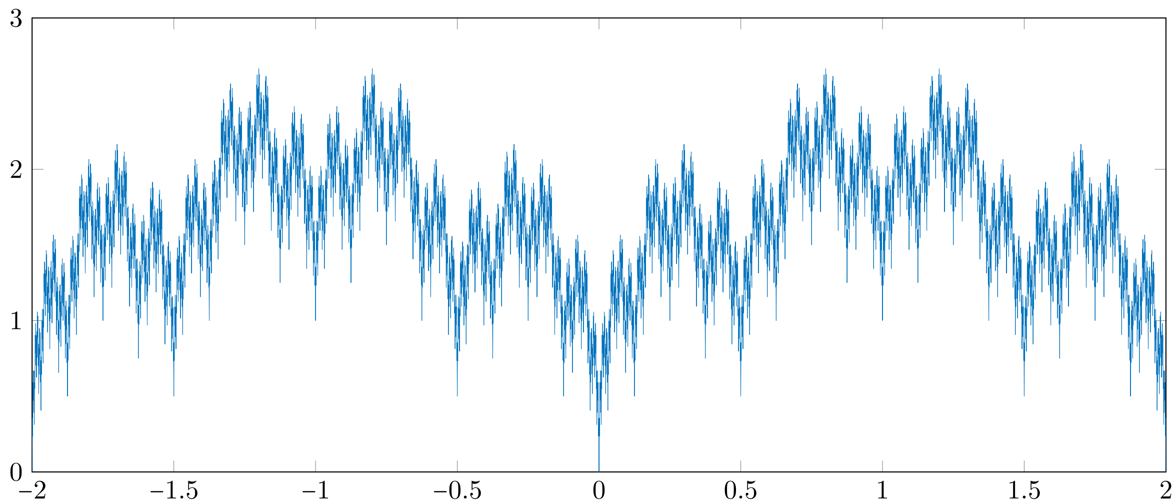
$$f(x) := \sum_{n=0}^{\infty} \varphi_n(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi_0(4^n x).$$

I have plotted $n = 1, 2, 3$, and $n = 5$; the graphs below are put in this order. (The one above is $n = 6$.)









Continuity and Differentiability

By the Weierstraß M-Test, each φ_n is bounded by $M_n := (3/4)^n$, so $\sum \varphi_n$ converges uniformly to f. Then, since each φ_n is continuous, Theorem 13.8.1 asserts that the limit f is **continuous**.

Now we show that f is **nowhere differentiable**. We pick $x \in \mathbb{R}$ and show that, by setting $\delta_m := \pm 4^{-m}/2$, the quotient

$$\frac{f(x+\delta_m) - f(x)}{\delta_m}$$

blows up (just like the 3^{-n} sequence in Devil's Staircase). The sign of δ_m is determined by requiring that no integer lies in the interval between $4^m x$ and $4^m(x+\delta_m) = 4^m x \pm 1/2$. (This is always possible!)
Next, we define

$$\gamma_n := \frac{\varphi_0(4^n(x+\delta_m)) - \varphi_0(4^n x)}{\delta_m}$$

(so the quotient $(f(x+\delta_m) - f(x))/\delta_m$ is just $\sum \gamma_n$). Now we bound γ accordingly:

(1) If $n > m$, then $4^n \delta_m = 4^{n-m}/2$ is an even integer. Since φ_0 is 2-periodic (i.e., periodic with period 2; check the first sawtooth graph), $\varphi_0(4^n x + 4^n \delta_m) = \varphi_0(4^n x)$, so the numerator = 0 and $\gamma_n = 0$.

(2) If $n = m$, $4^n \delta_m$ is precisely $1/2$, so in this case

$$|\gamma_n| = \left| \frac{1/2}{4^{-m}/2} \right| = 4^m.$$

(3) Finally, if $n < m$, notice that φ_0 is Lipschitz with Lipschitz constant = 1. Then,

$$|\varphi_0(4^n(x+\delta_m)) - \varphi_0(4^n x)| \leqslant |4^n(x+\delta_m) - 4^n x| = 4^n \delta_m.$$

Therefore $|\gamma_n| \leqslant 4^n \delta_m / \delta_m = 4^n$.

Finally, using (reverse) triangle inequality $|a + \sum b_i| \geqslant |a| - \sum |b_i|$, we have

$$\begin{aligned} \left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| &= \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right| \\ [\Delta] &\geqslant \left(\frac{3}{4}\right)^m \underbrace{|\gamma_m|}_{=4^m} - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \underbrace{|\gamma_n|}_{\leqslant 4^n} \\ &\geqslant 3^m - \sum_{n=0}^{m-1} 3^n = 3^m - \frac{1-3^m}{2} = \frac{3^m - 1}{2} \end{aligned}$$

which blows up as $n \to \infty$. Therefore $f'(x)$ exists nowhere, and we are done!! □