Math 425A Past Midterm

Quiz problems (No proofs are required in this part, unless specified otherwise)

(1) (1 pt) What is the supremum and the infimum of the set $A := \{m/n \colon m, n \in \mathbb{Z}, m \leq 5, n \geq 2\}$?

(2) (1 pt) Which of the sets

$$A := \left\{ \frac{x}{y} \colon x \in [0,1], y \in \mathbb{Q} \setminus \{0\} \right\},$$

$$B := \left\{ (x,y) \in \mathbb{R}^2 \colon x \in \mathbb{Q} \text{ and } y = \sqrt{2} + q \text{ for some } q \in \mathbb{Q} \right\}$$

is at most countable? Give a brief explanation (1 sentence).

(3) (1 pt) For which values of $\alpha \in \mathbb{R}$ does the sequence $(1/n^{2\alpha})_{n \geq 1}$ converge?

(4) (2 pts) Let $(X, d) := (\mathbb{R}, |\cdot|)$. Consider the set $\mathbb{Z} \subset \mathbb{R}$ of all integers. Is $\mathbb{Z}$

(a) open?

(b) closed?

(c) perfect?

(d) What are the limit points (if any) of $\mathbb{Z}$?

(e) What is the closure of $\mathbb{Z}^c$ (the complement of $\mathbb{Z}$)?

(5) (1 pt) Let $(X, d) := (\mathbb{R}, |\cdot|)$. Give an example of a subset of $\mathbb{R}$ that is perfect, but not compact.

(6) (2 pts) Let $(X, d) := (\mathbb{R}^2, |\cdot|)$ and let $K \subset \mathbb{R}^2$ be a bounded set with the following property: for any sequence $(x_n)_{n \geq 1} \subset K$ there exists a subsequence $(x_{n_k})_{k \geq 1}$ such that $x_{n_k} \overset{k \to \infty}{\longrightarrow} x$ for some $x \in \mathbb{R}^2$. Does this imply that K is compact? Give a brief explanation, or a counterexample.

(7) (1 pt) What are the limit points of the sequence $(x_n)_{n \geq 1} \subset \mathbb{R}$ given by $x_n := \cos(n\pi/3)$?

(8) (1 pt) Let $x_n := (-1)^n \frac{n+1}{n}$. Does (x_n) have a Cauchy subsequence? Is (x_n) Cauchy?

Proof problems

9. (4 pts) Show that

$$(x_1 y_1 + 3x_2 y_2 + 2x_3 y_3)^2 \leq \left(x_1^2 + 3x_2^2 + 2x_3^2\right)\left(y_1^2 + 3y_2^2 + 2y_3^2\right)$$

for any $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$.

10. (4 pts) Let $(X, d) := (\mathbb{R}, |\cdot|)$. Suppose that $x_n \overset{n \to \infty}{\longrightarrow} x$ for some $x \in \mathbb{R}$. Use the definition of convergence to show that

$$x_n + \frac{n+1}{n} \to x + 1 \quad \text{as } n \to \infty.$$

11. (4 pts) Let $(X, d) := (\mathbb{R}, |\cdot|)$, let $(x_n)_{n \geq 1} \subset \mathbb{R}$ be any sequence, and let

$$E := \left\{ y \in \mathbb{R} \cup \{\pm\infty\} \colon \text{ there exists a subsequence } (x_{n_k})_{k \geq 1} \text{ such that } x_{n_k} \overset{k \to \infty}{\longrightarrow} y \right\}$$

denote the set of all limit points of (x_n). Show that E is not empty.

12. (4 pts) Find the limit of the sequence $x_n := \sqrt[n]{5 + (-1)^n \cdot 2^{((-1)^{n+1})}}$.