Math 425A Midterm

Quiz problems (No proofs are required in this part, unless specified otherwise)

(1) (1 pt) What is the supremum and the infimum of the set $A := \{m/n : m, n \in \mathbb{Z}, m \leq 5, n \geq 2\}$? $\sup A = \frac{5}{2}$, $\inf A = -\infty$

(2) (1 pt) Which of the sets

$$A := \left\{ \frac{x}{y} : x \in [0,1], y \in \mathbb{Q} \setminus \{0\} \right\},$$

$$B := \left\{ (x,y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ and } y = \sqrt{2} + q \text{ for some } q \in \mathbb{Q} \right\}$$

is at most countable? Give a brief explanation (1 sentence). Only B is countable as $B = \underbrace{\bigcup_{x\in\mathbb{Q}} \bigcup_{q\in\mathbb{Q}} \{(x, \sqrt{2}+q)\}}_{\text{countable union of finite sets}}$ (A is uncountable as $A \supset \underbrace{[0,1]}_{\text{uncountable}}$)

(3) (1 pt) For which values of $\alpha \in \mathbb{R}$ does the sequence $(1/n^{2\alpha})_{n\geq 1}$ converge? $\alpha \geq 0$

(4) (2 pts) Let $(X,d) := (\mathbb{R}, |\cdot|)$. Consider the set $\mathbb{Z} \subset \mathbb{R}$ of all integers. Is $\mathbb{Z}$

(a) open? NO

(b) closed? YES

(c) perfect? NO

(d) What are the limit points (if any) of $\mathbb{Z}$? No limit pts

(e) What is the closure of $\mathbb{Z}^c$ (the complement of $\mathbb{Z}$)? $\overline{\mathbb{Z}^c} = \mathbb{R}$

(5) (1 pt) Let $(X,d) := (\mathbb{R}, |\cdot|)$. Give an example of a subset of $\mathbb{R}$ that is perfect, but not compact. $[0,\infty)$

(6) (2 pts) Let $(X,d) := (\mathbb{R}^2, |\cdot|)$ and let $K \subset \mathbb{R}^2$ be a bounded set with the following property: for any sequence $(x_n)_{n\geq 1} \subset K$ there exists a subsequence $(x_{n_k})_{k\geq 1}$ such that $x_{n_k} \overset{k\to\infty}{\longrightarrow} x$ for some $x \in \mathbb{R}^2$. Does this imply that K is compact? Give a brief explanation, or a counterexample. No, take $K := \{x \in \mathbb{R}^2 : |x| < 1\}$

(7) (1 pt) What are the limit points of the sequence $(x_n)_{n\geq 1} \subset \mathbb{R}$ given by $x_n := \cos(n\pi/3)$? $E = \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$

(8) (1 pt) Let $x_n := (-1)^n \frac{n+1}{n}$. Does (x_n) have a Cauchy subsequence? Is (x_n) Cauchy?

YES NO

Proof problems

9. (4 pts) Show that

$$(x_1y_1 + 3x_2y_2 + 2x_3y_3)^2 \leq (x_1^2 + 3x_2^2 + 2x_3^2)(y_1^2 + 3y_2^2 + 2y_3^2)$$

for any $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$. Apply the Cauchy-Schwarz inequality with $x := (x_1, \sqrt{3}x_2, \sqrt{2}x_3)$, $y := (y_1, \sqrt{3}y_2, \sqrt{2}y_3)$

10. (4 pts) Let $(X,d) := (\mathbb{R}, |\cdot|)$. Suppose that $x_n \overset{n\to\infty}{\longrightarrow} x$ for some $x \in \mathbb{R}$. Use the definition of convergence to show that

$$x_n + \frac{n+1}{n} \to x + 1 \quad \text{as } n \to \infty.$$

Let $\varepsilon > 0$. $\exists N$ s.t. $|x_n - x| \leq \frac{\varepsilon}{2}$ for $n \geq N$. So for $n \geq \max(N, \frac{2}{\varepsilon})$

$$\left|x_n + \tfrac{n+1}{n} - x - 1\right| = \left|x_n - x + \tfrac{1}{n}\right| \leq \underbrace{|x_n - x|}_{\leq \frac{\varepsilon}{2}} + \underbrace{\tfrac{1}{n}}_{\leq \frac{\varepsilon}{2}} \leq \varepsilon.$$

11. (4 pts) Let $(X,d) := (\mathbb{R}, |\cdot|)$, let $(x_n)_{n\geq 1} \subset \mathbb{R}$ be any sequence, and let

$$E := \left\{ y \in \mathbb{R} \cup \{\pm\infty\} : \text{ there exists a subsequence } (x_{n_k})_{k\geq 1} \text{ such that } x_{n_k} \overset{k\to\infty}{\longrightarrow} y \right\}$$

denote the set of all limit points of (x_n). Show that E is not empty.

Case 1 (x_n) is not bdd above. Then there $\exists n$ $x_n > M$

Pick any sequence of $M_k \to \infty$ to obtain a sequence $(x_{n_k})_{k\geq 1}$ s.t. $x_{n_k} > M_k$ $\forall k$. Thus $x_{n_k} \longrightarrow \infty$ as M_k does $\Rightarrow \infty \in E$

Case 2 (x_n) is not bdd below $\Rightarrow \exists (x_{n_k})_{k\geq 1}$ s.t. $x_{n_k} \longrightarrow -\infty \Rightarrow -\infty \in E$

Case 3 (x_n) is bdd above and below.

$\xRightarrow{\text{Bolzano-Weierstrass}} \exists (x_{n_k})_{k\geq 1}$ $x_{n_k} \longrightarrow x$ for some $x \in \mathbb{R} \Rightarrow x \in E$

$\Rightarrow E$ is not empty

12. (4 pts) Find the limit of the sequence $x_n := \sqrt[n]{5 + (-1)^n \cdot 2^{((-1)^{n+1})}}$.

Note

$$\sqrt[n]{5-2} \leq x_n \leq \sqrt[n]{5 + \tfrac{1}{2}} \quad \forall n$$

both $\xrightarrow{n\to\infty} 1$ (proved in class)

So $x_n \xrightarrow{n\to\infty} 1$ by the squeeze thm.