

Math 425A Exam

Quiz problems (No proofs are required in this part)

1. (2 pts) Consider the following subsets of \mathbb{R} :

$$A := \{e^{-n} : n \in \mathbb{N}\}, \quad B := [0, 1), \quad C := (0, 1/2) \cup (1/2, 1), \quad D := [0, 1] \setminus \mathbb{Q}.$$

Which of these sets are

- (a) closed?
- (b) open?
- (c) connected?
- (d) dense in $[0, 1]$?

2. (3 pts) Let (X, d) be a metric space, and $E \subset X$ be bounded.

- (a) Give an example of a metric space, which has infinitely many elements, in which the above implies that E is compact.
- (b) Give an example of a metric space in which this implies that E is precompact (i.e. its closure, \overline{E} , is compact), but not necessarily compact.
- (c) Give an example of a metric space in which this does not imply that E is precompact.

3. (4 pts) Let $f(x) := |x|^\alpha$ for $\alpha \geq 0$, $x \in \mathbb{R}$. For which values of α is f

- (a) a continuous function?
- (b) a uniformly continuous function?
- (c) a differentiable function?
- (d) Now consider $f: (0, \infty) \rightarrow \mathbb{R}$, defined by $f(x) := \sin(x^2)$. Is it a continuous/uniformly continuous function? Is it a differentiable function?

4. (2 pts) Consider the sequence $a_n := \cos(n\pi/2)$.

- (a) How many distinct limit points does $(a_n)_{n \geq 1}$ have?
- (b) What is $\liminf_{n \rightarrow \infty} a_n$ and $\limsup_{n \rightarrow \infty} a_n$?
- (c) Does the series $\sum_{n \geq 1} \frac{a_{2n}}{\sqrt{n}}$ converge?

5. (2 pts) Let (X, d) be a metric space, and $E \subset X$ be connected. Give an example of a metric space in which this means that E is path-connected, and another example where it does not mean that (give a counterexample).

6. (3 pts) Let $f_n(x) := x^n$ for $x \in [0, 1]$, $n \in \mathbb{N}$.

- (a) Does the series $\sum_{n \geq 0} f_n$ converge uniformly on $[0, 1]$?
- (b) How about on $[0, 1/2]$?
- (c) Is it true that $\int_0^{1/2} \sum_{n \geq 0} f_n(x) dx = \sum_{n \geq 0} \int_0^{1/2} f_n(x) dx$?

Proof problems

7. (4 pts) Let (K, d) be a compact metric space and B be the subset of the metric space $(C(K), \|\cdot\|_{\text{sup}})$, defined by

$$B := \{f \in C(K) : |f(x)| < 1 \text{ for all } x \in K\}.$$

Show that B is open.

8. Let E be a connected subset of a metric space (X, d) . Let $f: E \rightarrow \{0, 1\}$ be continuous. Show that f is constant.

9. (4 pts) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be three times differentiable. Suppose that $f(a) = f(b) = f'(a) = f'(b) = 0$ for some $a, b \in \mathbb{R}$, $a < b$. Show that there exists $c \in (a, b)$ such that $f'''(c) = 0$.

10. (3 pts) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f(-1) = f(1) = 0$ and $f'(x) \rightarrow \infty$ as $x \rightarrow \infty$. Show that there exists $x_0 \in \mathbb{R}$ such that $f'(x_0) = 2021$.

11. (3 pts) Let

$$f(x) := \begin{cases} 0 & x \in \mathbb{Q}, \\ x^2 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that f is not Riemann integrable on $[0, 1]$.

12. (4 pts) Let $f(x) := 3x + \sin x$. Show that there exists $z \in [0, \pi]$ such that $f(z) = 3$ in two ways: first using the Intermediate Value Theorem for continuous functions, and then using the Darboux Theorem.

13. (4 pts) Let $f_n, f \in C([0, 1])$ be such that $f_n \rightarrow f$ as $n \rightarrow \infty$ (in the sense of convergence in the metric space $(C([0, 1]), \|\cdot\|_{\text{sup}})$). Show that if

$$F_n(x) := \int_0^x f_n(t) dt$$

then $F_n \rightrightarrows F$ as $n \rightarrow \infty$ on $[0, 1]$, for some $F \in C([0, 1])$.

14. Let $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Does there exist a sequence of continuous functions on \mathbb{R} that converge uniformly to f ? (Give an example or prove that such a sequence does not exist.)

15. (5 pts) Given $x \in [1, 2]$ let

$$f(x) := \sum_{n \geq 1} (-1)^n e^{-nx}.$$

- (a) Is $f(x)$ well-defined?
- (b) Is f uniformly continuous?

16. (5 pts) Let (f_n) be a sequence of Riemann integrable functions on $[0, 1]$, such that for some $M > 0$, $|f_n(x)| \leq M$ for all $n \geq 1$, $x \in [0, 1]$. Consider

$$F_n(x) := \int_{\frac{x}{n}}^x f_n(t) dt$$

for $x \in [0, 1]$, $n \geq 1$. Show that there exists a subsequence $(F_{n_k})_{k \geq 1}$ that is Cauchy in $C([0, 1])$.