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1. a) ~~A, B~~ B

   b) None

   c) B, C

   d) B, C, D

2. a) $[0,1]$ on R, with standard Euclidean metric d.

   b) ~~$X := R$, $E := (0,\infty)$~~. $X := (R, d)$, $E := (0, \infty)$

   c) $(C(\cancel{K}\,[0,1]), \|\cdot\|_{sup})$, with $C(\cancel{K}\,[0,1])$ equicontinuous and uniformly bounded

   d) ~~$X := R$, E~~ ~~$X := Z$, $E := Z^+$~~. $X := (Z, d)$ $E := \{Z^+\}$

3. a) No

   b) No

4. a) 4

   b) $\liminf_{n\to\infty} a_{qn+1} = -\frac{1}{2}$, $\limsup_{n\to\infty} a_{qn+1} = \frac{1}{2}$

5. a) No

   b) ~~No~~ Yes

6. a) No

   b) Yes

   c) No

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7. Proof: We have $r_n \to \infty$ as $n \to \infty$. ~~Thus, $\forall y \in K$, $d(x,y) < r_n$ for some n.~~
It is trivial that $\{N_{r_n}(x)\}$ is an open cover of K, as neighborhoods are open.
$\because$ K is compact. $\therefore$ $\cancel{K \subset \bigcup_{n=1}^{N}}$ $K \subset \bigcup_{n=1}^{a} N_{r_n}(x)$ for some $a < \infty$, $a \in \mathbb{Z}^+$.
$\Rightarrow \forall y \in K$, $d(x,y) < \sum_{n=1}^{a} r_n$ as guaranteed, because a is finite $\Rightarrow \sum_{n=1}^{a} r_n$ also finite.
$\Rightarrow$ K is thus bounded.

8. Continuous function on compact domain attains its bounds. Thus $\exists\, x_0$ s.t. $g(x_0) = \sup g(x)$.
clearly $f(x) \le g(x) \Rightarrow \sup_{x \in X} |f(x)| \le \sup_{x \in X} |g(x)| = g(x_0)$. $\rightarrow$ some $f(x)$ s.t.
V is closed if it contains all its limit points, i.e. $\nexists$ $\sup_{x \in X} |f(x) - g(x)| < \varepsilon$ $\forall \varepsilon > 0$.
$\Rightarrow$ this requirement is satisfied ~~when~~ because ~~all $f(x) \le g(x)$~~ qualified $f(x)$ always $\le g$.
Hence V is closed

9. Suppose not, then $\exists\, y \in (a,b)$ s.t. $f(y) = c$. clearly $[x,y] \in$ differentiable, as $f \in D([a,b])$.
By MVT $\exists\, k \in (x,y)$ s.t. $f'(k) = \cancel{\frac{f(x)-f(y)}{}} \frac{f(y)-f(x)}{y-x} = \frac{c-c}{y-x} = 0$.
$\because f'(x) > 0$ on $(a,b) \Rightarrow f'(x) \ne 0$, contradiction. Hence $f(y) \ne c$.
$\Rightarrow \exists$ at most one $x \in (a,b)$ s.t. $f(x) = c$.

10. let $g(x) = f(x)^2$, $h(x) = x^2$. $\because f \in D(a,b)$ ~~$D[a,b]$~~, $\Rightarrow$ apply general MVT.
$$\frac{g'(x)}{h'(x)} = \frac{g(b)-g(a)}{h(b)-h(a)} = \frac{f(b)^2 - f(a)^2}{b^2 - a^2} = 1. \Rightarrow \frac{2f(x)f'(x)}{2x} = 1.$$
$\Rightarrow f(x)f'(x) = x$. Hence general MVT guarantees at least one solution in $[a,b]$.

11. We first assume a partition, $[x_1, x_2 \cdots x_n]$.
By definition, ~~$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i =$~~ $U(P,f) \ge \inf\left(\sum_{i=1}^{n} M_i \Delta x_i\right) = \sum_{i=1}^{n} 9 \cdot \Delta x_i = 9$.
$\Rightarrow$ we can do this because $\mathbb{Q}$ dense in $\mathbb{R}$ (hence any interval contains $q \in \mathbb{Q}$), and $\inf(10 + \sin x) = 9$.
Similarly, $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{R}$, and it's trivial that $L(P,f) \le \sup\left(\sum_{i=1}^{n} M_i \Delta x_i\right) = 0$.
Actually, $L(P,f) = 0 \cdots$ Take ~~$\varepsilon > 1$~~ $\varepsilon = 1$, $U(P,f) - L(P,f) \ge 9 > 1 = \varepsilon$.
$\Rightarrow$ This violates riemann integrability criteria.

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12. let $H(x) = F(x) - G(x)$. $H'(x) = F'(x) - G'(x) = 0$.

13. a) Yes. By comparison test, $\sum_{n\geq 1} \left|\frac{\cos((n^4+2)x)}{n^2}\right| \leq \sum_{n\geq 1} \left|\frac{1}{n^2}\right|$, which converges.

$\Rightarrow$ Hence $f(x)$ is well-defined.

b) Yes. Apply Weierstrass-M test, we see $\left|\frac{\cos((n^4+2)x)}{n^2}\right| \leq \frac{1}{n^2}$, and $\sum_{n\geq 1} \frac{1}{n^2}$ converges.

$\Rightarrow$ Hence the series $\sum_{n\geq 1} \frac{\cos((n^4+2)x)}{n^2}$ converges uniformly to f.

It's trivial that $\frac{\cos((n^4+2)x)}{n^2}$ is continuous. $\Rightarrow$ f is the uniform limit, must also be continu

$\Rightarrow$ f is continuous on $[0,1]$, a compact domain. Hence f is uniformly continuous.

14. $|f_n(x) - f_n(y)| \leq M|x-y|^{3/4} \Rightarrow |f_n(x) - f_n(y)| \leq M|x-y|$ $\forall x, y \in [0,1]$ as $M > 0$, $|x-y| > 0$.

Hence $f_n(x)$ must be lipschitz. This implies equicontinuity. $\Rightarrow \forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{M}$

immediately gives the claim. Also, $f_n(x)$ is bounded. We have $f_n(0) = 0$,

$\Rightarrow |f_n(x) - f_n(0)| \leq M|x - 0|$, and $\sup|x| = 1 \Rightarrow |f_n(x)| \leq M$. uniformly $\forall x \in [0,1]$, $n \geq 1$.

$\Rightarrow$ Two prerequisites for Arzela-Ascoli satisfied.

$\Rightarrow$ we say $\{f_n\}$ admits a uniformly convergent subsequence, which is exactly:

$\exists \{f_{n_k}\}_{k\geq 1}$ and $f \in C([0,1])$ s.t. $\sup_{x\in[0,1]} |f_{n_k}(x) - f(x)| \to 0$ as $n \to \infty$.

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↳ (Cont. 12) let $H(x) = F(x) - G(x)$. $H'(x) = F'(x) - G'(x)$. $\forall$

Also, $H'(x) = G'(x) - F'(x) = 0$. because $H'(x) = 0 \Rightarrow$ then $H(x) = H(y)$

Assume $\exists\, x, y \in (a,b)$, $H(x) \neq H(y) \overset{\text{by MVT}}{\Rightarrow} H'(z) = \frac{H(y)-H(x)}{y-x} \neq 0$. Violation. $\Rightarrow H(x) \equiv C$, $G(x) = F(x) + C$