Math 425A Problem Sheet 1 (due 9am on Monday, 6th Sep)

We use the convention that the natural numbers do not include 0, i.e. $\mathbb{N} := \{1, 2, \ldots\}$. Unless specified otherwise, you should assume that a given set is considered as a subset of $\mathbb{R}$.

Essential problems

1. (3 pts) Prove the claim of Example 1.17 for the field of rational numbers $\mathbb{Q}$ without using the notion of the real numbers. In other words show that if $x, y \in \mathbb{Q}$ are such that $x > 0$ then there exists $n \in \mathbb{N}$ such that $nx > y$, but do not use Example 1.17 in your solution. *(Comment: if you use some function (such as the floor function), you need to verify that it is well defined for rational numbers; namely you need to demonstrate how exactly is the output of the function computed.)*

2. (4 pts) Find the supremum and the infimum of

$$A := \left\{ \frac{(n+m)^2}{2^{nm}} : n, m \in \mathbb{N} \right\}.$$

3. (3 pts) Use induction to show that if a_1, ... a_n are positive real numbers such that $a_1 a_2 \ldots a_n = 1$ then

$$a_1 + a_2 + \ldots + a_n \geq n.$$

Additional problems

4. (1 pt) Show that

$$\max\{x, y\} = \frac{|x - y| + x + y}{2}.$$

Find a similar formula for $\min\{x, y\}$ (and prove it).

5. (1 pt) Find sup and inf of

$$A := \left\{ \frac{m}{m+n} : m, n \in \mathbb{N} \right\}.$$

6. (1 pt) Let $A \subset \mathbb{R}$ be bounded above. Show that then $-A := \{-x : x \in A\}$ is bounded below and

$$\inf(-A) = -\sup A.$$

7. (1 pt) Let $A, B \subset \mathbb{R}$ be bounded above. Show that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

8. (1 pt) Let $A, B \subset \mathbb{R}$ be bounded above and below. We set $A + B := \{a + b : a \in A, b \in B\}$ and $A - B := \{a - b : a \in A, b \in B\}$. Show that

$$\sup(A + B) = \sup A + \sup B$$

and

$$\sup(A - B) = \sup A - \inf B.$$

Is it true that

$$\inf(A + B) \leq \inf A + \inf B?$$

9. (1 pt) Let $A := [0, \infty)$, $B := \mathbb{N}$, each equipped with the usual notions of addition, multiplication and the order relation. Which of A, B have the least upper bound property? Which of A, B is a field?

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