MATH 425a HW 1

1\. ~~proof: let $n=\frac{a}{b}, m=\frac{c}{d}$, with $a,b,c,d>0$.~~

proof: By definition of $\mathbb{Q}$, $\exists a,b,c,d$ s.t. $x=\frac{a}{b}$, $y=\frac{c}{d}$, $a,b,c \in \mathbb{N}^+$, $d \in \mathbb{Z}$.

Consider two cases (in both cases, $x>0$).

1) $y \le 0$. Then pick $n=1$, since $y \le 0 < 1x$.

2) $y>0$. $\because c>0, d>0$, $\therefore y=\frac{c}{d} \le c \le bc(\frac{a}{b}) < (bc+1)x$ (since $b,c,d \in \mathbb{N}^+$). $n=bc+1$ positive.

$\therefore$ from 1) 2), $\forall x,y \in \mathbb{Q}, x>0, \exists n \in \mathbb{N}^+$ s.t. $nx>y$.

2\. Solution: ~~Note~~ ~~$\frac{(n+m+1)^2}{2nm} = \frac{(n+m)^2}{2nm}$~~ let $f(n,m) := \frac{(n+m)^2}{2nm}$

i) Note that $f(n+1,m)=\frac{(n+1+m)^2}{2(n+1)m} = \frac{(n+m)^2}{2nm}\cdot\frac{(n+m+1)^2}{(n+m)^2 2m} \le \frac{(n+m)^2}{2nm}\cdot\frac{(n+m+1)^2}{2(n+m)^2}$.

↳ The above conclusion is true because $n \in \mathbb{N}$.

for the last quotient, note that $x := (n+m+1)^2 - 2(n+m)^2 = 2(n+m)+1-(n+m)^2$.

It's obvious that $x<0$ when $n+m \ge 3$, simply by listing potential $\mathbb{N}$ values for n and m.

when $x<0$ ($n+m \ge 3$), $f(n,m+1) < f(n,m)$, $f(n+1,m) < f(n,m)$ because n,m are interchangable.

$\therefore$ the supremum of A ($f(n,m)$) cannot be achieved when $n+m \ge 3$.

1) let $n=m=1$. $f(n,m)=\frac{2^2}{2\cdot 1\cdot 1}=2$.

2) let $n=1, m=2$ (or $n=2, m=1$). $f(n,m)=\frac{(2+1)^2}{2\cdot 2\cdot 1}=\frac{9}{4}=2.25$. No other possibilities.

From 1) 2), supremum of A is 2.25.

ii) Now find infimum. From an initial observation $\inf(A)=0$.

1) $\forall n,m \in \mathbb{N}$, $\frac{(n+m)^2}{2nm}>0$. Therefore 0 is a lower bound for A.

2) Suppose 0 is not A's infimum. Then $\exists \varepsilon>0$ s.t. $\forall n,m \in \mathbb{N}$, $\frac{(n+m)^2}{2nm} \ge \varepsilon$.

It's obvious that when n,m get sufficiently large, $\frac{(n+m)^2}{2nm}$ will be less than ε (exponentials grow faster in the denominator.

from 1) 2) $\inf(A)=0$.

$\therefore$ from i) ii) $\sup(A)=2.25$, $\inf(A)=0$.

3\. Denote the statement ($a_1+a_2+a_3+\cdots+a_n \ge n$ s.t. $a_1a_2\cdots a_n=1$, if $a_1,a_2\cdots a_n \in \mathbb{R}^+$) by P.

when $n=1$, $a_1=1\ge 1 \Rightarrow P(1)$ True. (1)

Suppose $P(k)$ true, $k \in \mathbb{N}$. Then $a_1+a_2+\cdots+a_k \ge k$ s.t. $a_1a_2\cdots a_k=1$, a_1 to $a_k \in \mathbb{R}^+$.

when $n=k+1$, $a_1+a_2+\cdots+a_{k+1}=a_1+\cdots+a_k+a_{k+1} \ge k+a_{k+1}=k+\frac{1}{a_1a_2\cdots a_k}$.

$\because a_1$ to $a_k \in \mathbb{R}^+$, $\therefore \frac{1}{a_1a_2\cdots a_k}<1$. $a_1+a_2+\cdots+a_{k+1} \ge k+a_{k+1}>k+1$.

$\therefore P(k+1)$ true whenever $P(k)$ true. (2)

~~P(1) true, P~~ $\therefore$ from (1)(2), P is true.

4. Proof: 1) $\cancel{x>y}$ Suppose $\cancel{x\geq y}$ $x>y. \Rightarrow \max\{x,y\}=x.$

$$\frac{|x-y|+x+y}{2}=\frac{x-y+x+y}{2}=\frac{2x}{2}=x=\max\{x,y\}.$$

2) Suppose $x<y \Rightarrow \max\{x,y\}=y.$

$$\frac{|x-y|+x+y}{2}=\frac{y-x+x+y}{2}=\frac{2y}{2}=y=\max\{x,y\}.$$

$\therefore$ from 1)2), $\max\{x,y\}=\frac{|x-y|+x+y}{2}.$

$\min\{x,y\}=\frac{x+y-|x-y|}{2}.$

1) $x>y$, $\min\{x,y\}=y.$

$$\frac{x+y-|x-y|}{2}=\frac{x+y-x+y}{2}=\frac{2y}{2}=y.$$

2) $\cancel{x<y}$ $y>x$, $\min\{x,y\}=x.$

$$\frac{x+y-|x-y|}{2}=\frac{x+y-y+x}{2}=\frac{2x}{2}=x.$$

$\therefore$ from 1)2), $\min\{x,y\}=\frac{x+y-|x-y|}{2}$

5. let $f(m,n)=\frac{m}{m+n}$, $m,n\in\mathbb{N}$.

$\because \frac{m}{m+n}=1-\frac{n}{m+n}$, $\frac{n}{m+n}>0$, $\therefore \sup(A)=1.$ $\left(\lim_{m=k,\, n\to\infty}\frac{n}{m+n}=1\right)$ $k\in\mathbb{N}$ $\cancel{k\in\mathbb{R}}$

$\because \frac{n}{m+n}\to 1$ when keeping either fixed and the other sufficiently large,

$\therefore \inf(A)=1-1=0.$

6. Proof: $\because \forall x\in A, -x\in -A \Rightarrow -x\geq \inf(-A)$, and by arithmetic operation:

$x\leq -\inf(-A)$ for $x\in A \Rightarrow -\inf(-A)$ an upper bound for A.

let α be any A's upper bound. Then $\forall x\in A$, $x\leq\alpha \Rightarrow -x\geq-\alpha$ for $-x\in -A$.

However, ~~inf~~ $-\alpha$ is a lower bound for $-x\in -A$. Then

$\inf(-A)\geq-\alpha$, $\alpha\geq \cancel{\inf} -\inf(-A)$ ~~for $x\in A$~~. ~~so~~ Any upper bound $\geq -\inf(-A)$.

$\therefore \sup(A)=-\inf(-A)$, $\inf(A)=-\sup(-A)$.

7. Proof: let $x=\sup(A\cup B)$. Then $x\geq a\ \forall a\in A$, $x\geq b$ ~~for~~ $\forall b\in B$.

Also, it follows that $x\geq\sup(A)$, $x\geq\sup(B)$.

Suppose $\sup(A)>\sup(B)$. Then $\max\{\sup(A),\sup(B)\}=\sup(A)$.

$\hookrightarrow \begin{cases} \cancel{\Rightarrow}\ \sup(\cancel{a}^{A})\geq a\ \cancel{\forall}\ \forall a\in A \\ \sup(\cancel{a}^{A})\geq b\ \forall b\in B \text{ because } \sup(A)\cancel{\geq}>\sup(B)\geq b. \end{cases}$

let $C\in A\cup B$. Then $\sup(C)=\sup(A)$ $\forall C\in A\cup B$. then $\nearrow \sup(C)=\sup(B)$.

Note that the following proof can be exactly reversed for $\sup(A)<\sup(B)$

$\therefore \sup(A\cup B)=\max\{\sup(A),\sup(B)\}$.

8、 i) Proof: let $r=a+b$. Note that $a \le sup(A)$ for $a\in A$, $b \le sup(B)$ for $b\in B$ ∴

$\because r \le sup(A)+sup(B) \Rightarrow sup(A)+sup(B)$ is an upper bound for $A+B$. $Sup(A+B) \le sup(A)+sup(B)$

Suppose $sup(A+B) \ne sup(A)+sup(B)$. Then $sup(A+B) < sup(A)+sup(B)$. Denote $sup(A+B)$ by μ.

Thus, $\exists\, \varepsilon>0$ s.t. $sup(A)+sup(B)-\mu=\varepsilon$.

Also, $\exists\, a\in A$, $sup(A)-a<(\frac{\varepsilon}{2})$ (Definition of supremum, + no difference between $\frac{\varepsilon}{2}$ and ε) when $\varepsilon \to 0$

$\exists\, b\in B$, $sup(B)-b<(\frac{\varepsilon}{2})$. Add these two equations,

$sup(A)+sup(B)-(a+b)<\varepsilon \Rightarrow \mu < a+b \Rightarrow \mu$ cannot be supremum of $(A+B)$. Contradiction.

Thus $sup(A+B)=sup(A)+sup(B)$.

ii) proof: let $q=a-b$. Repeat the above process. $sup(A-B) \le sup(A)-$ ~~sup~~ $inf(B)$.

↳ because $a \le sup(A)$ $\forall a\in A$, $b>inf(B)$ $\forall b\in B$. Denote $sup(A-B)$ by σ.

Suppose $sup(A-B) \ne sup(A)-inf(B)$. $\Rightarrow sup(A-B)<sup(A)-inf(B)$. Then:

$$\begin{cases} ① \ \exists\, \varepsilon>0 \text{ s.t. } sup(A)-inf(B)-\sigma=\varepsilon. \\ ② \ \exists\, a\in A,\ sup(A)-a<\frac{\varepsilon}{2} \\ ③ \ \exists\, b\in B,\ \cancel{b-inf(B)<\frac{\varepsilon}{2}}\ inf(B)-b<-\frac{\varepsilon}{2} \end{cases}$$

② - ③ yields $sup(A)-inf(B)-(a-b)<\varepsilon \Rightarrow \sigma<a-b \Rightarrow \sigma$ cannot be $sup(A-B)$. Contradiction.

Thus $sup(A-B)=sup(A)-inf(B)$.

iii) This statement is false.

proof: let $\gamma=a+b$, $a\in A$, $b\in B$. Then $a \ge inf(A)$, $b \le inf(B)$.

$\gamma \ge inf(A)+inf(B) \Rightarrow inf(A)+inf(B)$ is a lower bound for $A+B$.

Thus $inf(A+B) \ge inf(A)+inf(B)$. The original argument is therefore false.

9、 Both A, B have LUBP as they are subsets of $\mathbb{R}$. ($\mathbb{R}$ has LUBP, taken for granted).

Neither is a field. ~~For field A~~ ~~Set A~~ $\forall x\in A$, $\nexists (-x)$ s.t. $x+(-x)=0 \Rightarrow -x$ will be negative, not in A.

$\forall y\in B$, $\nexists (\frac{1}{y})$ s.t. $y\cdot(\frac{1}{y})=1 \Rightarrow \frac{1}{y}$ must not be an integer, not in B.