#1

Lemma. For every positive rational number $r \in \mathbf{Q}$, there exists unique positive integers p and q such that p and q are coprime and $r = p/q$.

Proof. We will take the *existence* of such a fraction for granted (divide the denominator and the numerator by the greatest common divisor) and will show that such a fraction is unique.

If α/β is another fraction of positive coprime integers such that $p/q = \alpha/\beta$, then we have

$$\frac{p}{q} = \frac{\alpha}{\beta} \iff p\beta = q\alpha \iff p \mid \alpha, \alpha \mid p \text{ and } q \mid \beta, \beta \mid q \iff p = \alpha \text{ and } q = \beta.$$

Hence there exists a unique fraction of positive coprime integers p/q that is equal to r. ■

If $y \leq 0$, there is nothing to be done: choose $n = 1$. Assume $y > 0$. If $x \geq y$, then again choose $n = 1$. So suppose the case where $y > 0$, and $x < y$, which is if and only if $1 < y/x$. Consider the following function $F : \mathbf{Q}^+ \to \mathbf{Z}^+$:

$$F(r) = a + 1,$$

where a is the numerator of the unique fraction of positive coprime integers a/b such that $a/b = r$. Then

$$F(y/x) - \frac{y}{x} = (a+1) - \frac{a}{b} = \frac{(a+1)b - a}{b} = \frac{ab + b - a}{b} = \frac{a(b-1) + b}{b}.$$

where a/b is of course the unique fraction of positive coprime integers that equals r. Note that since b is a positive integer, $a(b-1)+b \geq b$, hence $F(y/x) - y/x \geq 1$, which implies $F(y/x) > y/x$, or equivalently, $y < F(y/x)x$. Hence the number we seek is $F(y/x)$ (or anything greater than that).

#2

For $n, m < 3$, there are only four points to check, and the largest value achievable there is 9/4 at $(1, 2)$ and $(2, 1)$. We will show that at all other points where $n, m \geq 3$, the value $(n+m)^2/2^{nm}$ is bounded above by something far less than 9/4. Note that if $a \in A$ is an upper bound, then it must be the supremum because any $b < a$ is not an upper bound. Hence if we show that 9/4 is an upper bound, then we will have proven that it is the supremum.

Observe that when $n, m \geq 3$,

$$\begin{aligned}\frac{(n+m)^2}{2^{nm}} &= \frac{n^2+2nm+m^2}{2^{nm}} \\ &= \frac{n^2}{2^{nm}} + \frac{2nm}{2^{nm}} + \frac{m^2}{2^{nm}} \\ &\leq \frac{n^2}{2^{3n}} + \frac{2nm}{2^{nm}} + \frac{m^2}{2^{3m}} \\ &= \frac{n^2}{8^n} + \frac{2nm}{2^{nm}} + \frac{m^2}{8^m}.\end{aligned}$$

We will show that each fraction is less than or equal to 9/512 when $n, m \geq 3$, therefore $(n+m)^2/2^{nm} < 27/512$ for all $n, m \geq 3$. To do that, it suffices to show that for all $j \geq 3$, $j^2/8^j < 9/512$ and for all $k \geq 9$, $k/2^k < 9/512$. Observe that if we put $a(j) := j^2/8^j$, then

$$a(j) - a(j+1) = \frac{(j+1)^2}{8^{j+1}} - \frac{j^2}{8^j} = \frac{(j+1)^2 - 8j^2}{8^{j+1}} = \frac{-7j^2+2j+1}{8^{j+1}}$$

Observe that the numerator factors as follows (you may need to use the quadratic formula to get it, but you can trivially verify the result by actually expanding it out):

$$-7j^2+2j+1 = -7\left(j - \frac{1-2\sqrt{2}}{7}\right)\left(j - \frac{1+2\sqrt{2}}{7}\right)$$

This quadratic expression will be negative if $j > (1+2\sqrt{2})/7$, and since $2 > \sqrt{2}$, for all $j > 5/7$, this expression is negative. This in turn implies that for every $j \geq 1$, $a(j) > a(j+1)$, so for every integer $j \geq 3$, $a(j)$ is bounded above by $a(3) = 9/512$.

Similarly, if we put $b(k) := k/2^k$, we get

$$b(k) - b(k+1) = \frac{k+1}{2^{k+1}} - \frac{k}{2^k} = \frac{k+1-2k}{2^{k+1}} = \frac{1-k}{2^{k+1}},$$

and evidently this is negative when $k \geq 2$. Hence for all $k \geq 9$, $b(k)$ is bounded above by $b(9) = 9/512$. Hence $\sup A = 9/4$.

We now claim that $\inf A = 0$. To show that, first observe that every $(n+m)^2/2^{nm}$ is positive, hence A is bounded below by 0, meaning 0 is a lower bound. To show that the second property of infimum is satisfied, let $\varepsilon > 0$ be given. Observe that when we take $n = m$, we obtain

$$\frac{(n+m)^2}{2^{nm}} = \frac{4n^2}{2^{n^2}}.$$

So if we can show that there exists a positive integer j such that $j/2^j < \varepsilon/4$, then we will have proven that 0 is the infimum by definition. Indeed, observe that

$$\frac{j}{2^j} < \frac{1}{j} \iff \frac{j}{2^j} - \frac{1}{j} = \frac{j^2 - 2^j}{j2^j} < 0 \iff j^2 - 2^j < 0.$$

Put $c(j) := j^2 - 2^j$. We will show that for every $j \geq 5$, $c(j) < -1$, therefore $c(j) < 0$.

For $j = 5$, $c(5) = -7 < -1$, so it is true. If we assume the result holds for $j = n$, then for $j = n+1$, we have

$$\begin{aligned} c(n+1) &= (n+1)^2 - 2^{n+1} \\ &= n^2 + 2n + 1 - 2^{n+1} \\ &< 2n^2 - 2 \cdot 2^n + 1 \\ &= 2(n^2 - 2^n) + 1 \\ &< -1. \end{aligned}$$

In getting the first inequality, we used the fact that when $n > 2$, $n^2 > 2n$. Thus for every integer $n > 2$, we get the inequality

$$\frac{4n^2}{2^{n^2}} < \frac{4}{n^2} = 4m^{-1}$$

for a positive integer $m = n^2$. So by the Archimedean property, if we set $y = 1/\varepsilon$ and $x = 1/4$, we are guaranteed a positive integer m such that $0 < 1/\varepsilon < m/4$, or equivalently (by L1.13), $4m^{-1} < \varepsilon$.

#3

We argue by induction. The base case when $n = 1$ is evident, so assume that the result holds for some n. Then we wish to show that if $a_1 a_2 \cdots a_n a_{n+1} = 1$, then $a_1 + a_2 + \cdots + a_n + a_{n+1} \geq n + 1$.

If all of a_i's are 1, then the result holds. If not, then since $a_1 a_2 \cdots a_n a_{n+1} = 1$, there exist at least one $a_i > 1$ and $a_j < 1$. Assume without loss that $a_i = a_n$ and $a_j = a_{n+1}$. Then $a_1 a_2 \cdots a_{n-1}(a_n a_{n+1}) = 1$, and by the induction hypothesis,

$$a_1 + a_2 + \cdots + a_{n-1} + (a_n a_{n-1}) \geq n \iff a_1 + a_2 + \cdots + a_{n-1} + (a_n a_{n-1} + 1) \geq n + 1.$$

So if we can somehow show that $a_n a_{n-1} + 1 \leq a_n + a_{n-1}$, we are done by transitivity. Indeed,

$$a_n a_{n-1} + 1 \leq a_n + a_{n-1} \iff a_n a_{n-1} + 1 - a_n - a_{n-1} \leq 0 \iff (1 - a_n)(1 - a_{n+1}) \leq 0,$$

and because $1 - a_n < 0$ and $1 - a_{n+1} > 0$, we have the inequality.

#4

Suppose $x > y$. Then $\max\{x, y\}$ must equal x, and indeed,

$$\frac{|x-y|+x+y}{2} = \frac{x-y+x+y}{2} = x.$$

If $y > x$, then $\max\{x, y\}$ must equal y, and

$$\frac{|x-y|+x+y}{2} = \frac{y-x+x+y}{2} = y.$$

If $x = y$, then either is the maximum, and

$$\frac{|x-y|+x+y}{2} = \frac{x-x+x+x}{2} = x = y.$$

For the minimum, we claim that that $\min\{x, y\} = -\max\{-x, -y\}$. Indeed, if $x \geq y$, then $-x \leq -y$. Hence

$$\min\{x, y\} = -\frac{|(-x)-(-y)|+(-x)+(-y)}{2} = \frac{x+y-|x-y|}{2}.$$

#5

We first claim that $\inf A = 0$. 0 is evidently a lower bound, and so consider some $\varepsilon > 0$.

Recall that by the Archimedean property, for any given rational numbers $x, y > 0$, there exists a positive integer k such that $y < kx$. If we take $x = 1$ and $y = 1/\varepsilon$, then we get a positive integer k such that $0 < y < k \iff 0 < k^{-1} < \varepsilon$. So choose $m = 1$ and $n = k - 1$ (we may insist that $k > 1$ to make it work). So ε cannot be a lower bound, and this shows that $\inf A = 0$.

We now claim that $\sup A = 1$. Evidently it is an upper bound, because $m < m + n$ for every $m, n \geq 1$. Let $u \in \mathbf{R}$ be a number such that $u < 1$. If $u < 0$, it clearly is not an upper bound, so assume $0 < u < 1$.

Let $\varepsilon > 0$ be such that $\varepsilon < 1 - u$. Denote by $f(m, n)$ the value $m/(m + n)$. Let μ be a positive integer such that $1/(1 + \mu) < \varepsilon$, whose existence is guaranteed by the Archimedean property. Then note that $1/(1 + \mu) = 1 - f(\mu, 1) < \varepsilon$. Consequently,

$$1 - f(\mu, 1) < \varepsilon < 1 - u \implies u < f(\mu, 1),$$

so u is not an upper bound. This shows that 1 is indeed the supremum.

#6

Observe first that for every $x \in A$, $x \leq \sup A \implies -\sup A \leq -x$, hence $-\sup A$ is indeed a lower bound for $-A$.

Let $-s \in \mathbf{R}$ be such that $-\sup A < -s \iff s < \sup A$. By the definition of $\sup A$, s cannot be an upper bound of A, meaning there exists some $a \in A$ such that $a > s$. Then $-a < -s$, i.e., $-s$ is not a lower bound of $-A$. This shows that $\inf(-A) = -\sup A$.

#7

Lemma. Let A be a set that has a supremum. Then for any upper bound u of A, $\sup A \leq u$.

Proof. Any upper bound u of A cannot be strictly less than $\sup A$ by definition, hence by trichotomy, $\sup A \leq u$. ∎

Let $m = \max\{\sup A, \sup B\}$, and let $x \in A \cup B$ arbitrary. If $x \in A$, then $x \leq \sup A \leq m$, and if $x \in B$, then $x \leq \sup B \leq m$, therefore $x \leq m$. This establishes $\sup(A \cup B) \leq \max\{\sup A, \sup B\}$.

To show the reverse inequality, observe that $\sup(A \cup B)$ is an upper bound of $A \cup B$. I.e., $\sup(A \cup B)$ is not only an upper bound of A, but also it is an upper bound of B. Therefore $\sup A \leq \sup(A \cup B)$ and $\sup B \leq \sup(A \cup B)$, hence $\max\{\sup A, \sup B\} \leq \sup(A \cup B)$. By trichotomy, we have $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

#8

Lemma. Let $A \subset \mathbf{R}$ be a set that has a supremum. Then for every $\varepsilon > 0$, there exists $a \in A$ such that $\sup A - a < \varepsilon$.

Proof. Suppose for contradiction that there exists $\varepsilon > 0$ such that for every $a \in A$, $\sup A - a \geq \varepsilon$. Then if we take $\sup A - \varepsilon$, this is strictly less than $\sup A$ yet $\sup A - \varepsilon - a \geq 0$ for every $a \in A$, i.e., $\sup A - \varepsilon$ is an upper bound, and this is a contradiction. ∎

Every element of $A + B$ is of the form $a + b$ for some $a \in A$ and $b \in B$. Note that $a \leq \sup A$ and $b \leq \sup B$, so $a + b \leq \sup A + \sup B$. Thus $\sup A + \sup B$ is an upper bound of $A + B$.

Let u be such that $u < \sup A + \sup B$. Then there exists some $\varepsilon > 0$ such that $\sup A + \sup B - u > \varepsilon$. By the lemma, there exist $a \in A$ and $b \in B$ such that $\sup A - a < \varepsilon/2$ and $\sup B - b < \varepsilon/2$. Therefore

$$\sup A + \sup B - u > \varepsilon \iff (\sup A - a) + (\sup B - b) + (a + b) - u > \varepsilon.$$

This leads to

$$\varepsilon < (a + b) - u + (\sup A - a) + (\sup B - b) < (a + b) - u + \varepsilon \implies u < a + b.$$

Hence u is not an upper bound, therefore $\sup A + \sup B = \sup(A + B)$.

Recall that in #6, we showed $\inf(-X) = -\sup X$. If we put $Y := -X$, then $\inf(Y) = -\sup(-Y)$. Thus

$$\sup(A - B) = \sup(A + (-B)) = \sup A + \sup(-B) = \sup A - \inf B.$$

Again using the fact that $\inf(-A) = -\sup A$, as well as the previous result, we obtain

$$\inf(-(-A - B)) = -\sup(-A - B)) = -\sup(-A) + \inf B = \inf A + \inf B.$$

#9

Let $E \subset A$ be any nonempty bounded (above) subset. Since $E \subset A \subset \mathbf{R}$, by the least-upper-bound-property of $\mathbf{R}$, E has a supremum in $\mathbf{R}$. Since every element of E is non-negative, its upper bound is non-negative, therefore it's supremum is non-negative. Hence the supremum lies in A, and so A has the least-upper-bound-property.

Let $E \subset \mathbf{N}$ be any nonempty bounded (above) subset. Since $\mathbf{N}$ is bounded below by 0, E too, is bounded below by 0. If $n \in \mathbf{N}$ is an upper bound of E, then E certainly contains less than $n+1$ many numbers, therefore any bounded above subset E of $\mathbf{N}$ is finite. The following lemma finishes the proof that $\mathbf{N}$ has the least-upper-bound-property.[1]

Lemma. A finite subset of an ordered set always contains its supremum.

Proof. We induct on the cardinality of a subset A. For every subset A of cardinality 1 (i.e., a singleton), the single element itself is the supremum. Assume the result holds for some cardinality $n > 1$, and suppose A is now a subset with $|A| = n+1$.

Let $\{x, y\} \in A$ be any two-element subset of A. Then we have[2] $\sup\{x, y\} \in \{x, y\}$. Assume without loss of generality that $\sup\{x, y\} = x$. Observe that $A \setminus \{y\}$ is a set with cardinality n, hence by the induction hypothesis, it attains the supremum, which we denote by s. We now use this to show that $\sup A$ exists and $\sup A = s$.

Since $s \in A \setminus \{y\} \subset A$, we have $s \in A$. Since s is the supremum of $A \setminus \{y\}$, for every $a \in A \setminus \{y\}$, we have $s \geq a$. Observe that to show that $\sup A = s$, what remains to show is that $s \geq y$. And indeed, since $s = \sup\{x, y\}$, we have $s \geq y$ as desired. ■

Neither is a field though, since both A and B lack the additive inverse of 1.

[1] In fact, what we show below is stronger than what we need show in that the following lemma not only asserts that a supremum exists, but it exists in the subset.

[2] You can either show this by doing strong induction (which, regardless of the word strong, is equivalent to ordinary induction) or by arguing that one of them is an upper bound, which is in the set, therefore by the fact we proved in #2, the supremum is contained.