Math 425A Problem Sheet 10 (due 9am on Monday, 8th Nov)

Essential problems

1. (2 pts) Suppose that $f \in C^2((-1,1))$ is such that $f(0)=0$, and let $g\colon (-1,1)\setminus\{0\} \to \mathbb{R}$ be defined by

$$g(x) := \frac{f(x)}{x}.$$

Show that one can extend g (at $x=0$) to a function that is $C^1((-1,1))$.

2. (3 pts) Let $f \in C^\infty((-2,2))$ (i.e. $f \in C^k((-2,2))$ for every $k \geq 0$) and suppose that $f(x)=0$ for infinitely many x's from the interval $[-1,1]$. Suppose also that $\sup_{(-2,2)} |f^{(n)}| = O(n!)$ as $n \to \infty$ (i.e. that there exists a constant $C>0$ such that for sufficiently large n we have $|f^{(n)}(x)| \leq C\, n!$ for all $x \in (-2,2)$). Prove that there exists an open subinterval of $(-2,2)$ where f vanishes (i.e. $f=0$ on a subinterval).

3. (2 pts) Suppose that a function $f\colon [a,b] \to \mathbb{R}$ has a local expansion at $x_0 \in (a,b)$ of order $n \geq 1$, i.e. that

$$f(x) = a_0 + a_1(x-x_0) + \ldots + a_n(x-x_0)^n + o((x-x_0)^n)$$

for some $a_0, a_1, \ldots, a_n \in \mathbb{R}$.

(a) Show that f is differentiable at x_0 and that $f(x_0)=a_0$, $f'(x_0)=a_1$.

(b) Consider $f(x) := x^3 \cos\frac{1}{x}$ for $x \neq 0$ and $f(0) := 0$, and show that it has a local expansion of order 2 at $x_0 := 0$ but that $a_2 \neq f''(x_0)/2$. *(Comment: this verifies the comment at Lem. 10.8.)*

4. (3 pts) Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x \in [0,1]\cap\mathbb{Q}, \\ 0 & \text{if } x \in [0,1]\setminus\mathbb{Q}. \end{cases}$$

Find $\overline{\int_0^1} f(x)\mathrm{d}x$ and $\underline{\int_0^1} f(x)\mathrm{d}x$ (i.e. the upper and lower Riemann integrals of f). Is f Riemann integrable on $[0,1]$? *(Hint: you might like to use the inequality $2ab \leq a^2+b^2$.)*

Additional problems

5. (1 pt) Determine whether each of the following limits exist and if yes then find the limit. Moreover, explain whether or not one can use de l'Hôpital rule to compute them.

(a) $\lim_{x\to\infty} \frac{x-\sin x}{2x+\sin x}$,

(b) $\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$ *(Hint: note that $f(x) = \mathrm{e}^{\log f(x)}$ for every $f(x)$.)*,

(c) $\lim_{x\to\infty} \frac{2x+\sin 2x+1}{(2x+\sin 2x)(\sin x+3)^2}$.

6. (1 pt) Let $f\colon (a,b) \to \mathbb{R}$ be such that $f''(x_0)$ exists for some $x_0 \in (a,b)$. Show that then

$$f''(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}.$$

7. (1 pt) Show that one can approximate the function $f(x) := \sqrt{1+x}$ on $(-1/2,1/2)$ by $1+\frac{x}{2}-\frac{x^2}{8}$ with the error of the approximation at most $|x|^3/2$ for every $x \in (-1/2,1/2)$.

8. (1 pt) Let $x > -1$ be such that $x \neq 0$. Show that $(1+x)^\alpha > 1 + \alpha x$ if $\alpha > 1$ or $\alpha < 0$, and that the inequality is opposite in the case when $\alpha \in (0,1)$. *(Comment: this is a generalisation of the statement that $(1+x)^n > 1 + nx$ (for $n \in \mathbb{N}$, $x > 0$), which you might have seen before (in fact we have used it in Ex. 5.1) and which can be proved directly using the binomial expansion.)*

9. (1 pt) Suppose that $f \in C^2((0,\infty))$ and that

$$\lim_{x\to\infty} xf(x) = \lim_{x\to\infty} xf''(x) = 0.$$

Show that $\lim_{x\to\infty} xf'(x) = 0$. *(Hint: Write $f(x+1)$ using Taylor's expansion at x.)*

10. (1 pt) Suppose that f and g have local expansions of order 2 at $x_0 \in \mathbb{R}$, i.e.

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + o((x - x_0)^2)$$
$$g(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + o((x - x_0)^2)$$

for some $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$. Show that then fg has a local expansion of order 2 at x_0 and find the coefficients.

11. (1 pt) Consider $f(x) := x^2$ defined on $[0,1]$ and

$$P := (x_k)_{k=0}^4, \qquad \text{where } x_k := \frac{k}{4}.$$

Show that P is a partition of $[0,1]$. Find $U(P,f,\alpha)$ and $L(P,f,\alpha)$ for $\alpha(x) := x - 1$.

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