MATH 425a PS 10

1. Define $h(x) = \begin{cases} f(x)/x & x \in (-1,1)\setminus\{0\} \\ f'(0) = \lim\limits_{t\to 0} \frac{f(t)-f(0)}{t-0} & x \in \{0\} \end{cases}$

First, we prove $h(x)$ differentiable.

① when $x \neq 0$, $h'(x) = \lim\limits_{t\to x} \frac{f(t)/t - f(x)/x}{t-x}$. Use L'H.

$= \lim\limits_{t\to x} \frac{1}{t^2}(tf'(t) - f(t)) = \frac{1}{x^2}\cdot(xf'(x) - f(x))$.

② When $x=0$, $h'(0) = \lim\limits_{t\to 0} \frac{h(t)-h(0)}{t-0} = \lim\limits_{t\to 0} \frac{f(t)/t - f'(0)}{t} \cancel{= \lim\limits_{t\to 0} \frac{f(t)/t}{t}}$ ($\frac{0}{0}$ case).

$= \lim\limits_{t\to 0} \frac{1}{t^2}(tf'(t) - f(t))$ $f'(t)$ bounded as $f \in C^2(-1,1)$.

$\hookrightarrow = \lim\limits_{t\to 0} \frac{tf''(t) + f'(t) - f'(t)}{2t} = \lim\limits_{t\to 0} \frac{f''(t)}{2} \Rightarrow = \frac{f''(0)}{2}$

$f''(0)$ exists as $f \in C^2(-1,1)$.

Thus $h(x)$ is differentiable.

We want to show $\cancel{\lim_{h\to}}$ $\lim\limits_{t\to 0} h'(t) = h'(0)$. (continuity definition).

we see $h'(0) = \underbrace{\lim\limits_{t\to 0} \frac{1}{t^2}(tf'(t) - f(t))}_{\uparrow \text{ from ②}} = \underbrace{\lim\limits_{t\to 0} h(t)}_{\uparrow \text{ from ①}} \Rightarrow h'(t)$ continuous at $x=0$

It's trivial that $h(x)$ continuous everywhere else. Thus ~~the~~ $h(x) \in C^1(-1,1)$.

2. $\because [0,1]$ compact, $\{x \mid f(x)=0\}$ has a subsequence $\{x_n^{(1)}\}$ converging to $x_n \in [-1,1]$.

$f(x_0)=0$ by continuity. Now, say between $x_n^{(1)}, x_{n+1}^{(1)}$, there's a $x_n^{(2)}$ s.t. $f'(x_n^{(2)}) = 0$ (Rolle's theorem). Also, $f'(x_0)=0$ by continuity.

$\hookrightarrow$ Repeat inductively, $f^{(k)}(x_0) = 0\ \forall k \in \mathbb{N}$.

$\sup |f^{(n)}| = O(n!) \Rightarrow \exists N \in \mathbb{N}, C>0$ s.t. $|f^{(n)}(x)| \le Cn!\ \forall x \in (-2,2)$, and $\forall y$ s.t. $|y - x_0| < 1$, $m \ge N$,

$\Rightarrow f(y) = f(x_0) + f'(x_0)(y-x_0) + \cdots * = \underbrace{\cancel{\sum_{j=0}^{m-1} \frac{f^{(j)}(x_0)}{j!}(y-x_0)^j}}_{\text{all } 0} + \sum_{j=m}^{\infty} \frac{f^{(j)}(x_0)}{j!}(y-x_0)^j$

Thus, ~~$f(y)$~~ $|f(y)| \le \frac{C}{1-(y-x_0)}(y-x_0)^m \Rightarrow$ when $m \to \infty$, $|f(y)| = 0$. $f(y)$ vanishes on $(x_0 - 1, x_0 + 1)$.

RHS $\to 0$, so

3. ~~lim~~

a) $$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{f(x)-a_0}{x-x_0}$$ ($f(x_0)=a_0$ trivial as $o((x-x_0)^n)$ would vanish.)

$$=\lim_{x\to x_0}\frac{a_1(x-x_0)+\cdots+a_n(x-x_0)^n}{x-x_0}$$

$$=\lim_{x\to x_0}(a_1+\cdots+a_n(x-x_0)^{n-1})=a_1+0+\cdots+0=a_1.$$

Thus f is differentiable at x_0.

b) $f'(x)=x^3(-\frac{1}{x^2})(-\sin\frac{1}{x})=x\sin\frac{1}{x}$ ~~(continuous as $x\neq 0$)~~.

$f''(x)=\sin\frac{1}{x}+x(-\frac{1}{x^2})\cos\frac{1}{x}=\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}$ ~~(this always exists if $x\neq 0$)~~.

~~Thus f(x) admits a second order expansion~~

We know $\lim_{x\to 0}\frac{x^3\cos(\frac{1}{x})}{x^2}=\lim_{x\to 0}x\cos(\frac{1}{x})=0$. Thus $f(x)=o((x-x_0)^2)$ at $x_0=0$.

↳ Second order expansion: $0+0(x-x_0)+0(x-x_0)^2+o((x-x_0)^2)$ at $x_0=0$.

~~lim~~ $f''(x_0)=\lim_{x\to x_0}\frac{f'(x)-f'(x_0)}{x-x_0}=\lim_{x\to 0}\frac{f'(x)-f'(0)}{x}$

⇓

~~$f'(x_0)$~~ $f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}$. We see x is not defined at 0. Thus $f''(0)$ does ~~exist~~ not exist.

Hence $a_2\neq\frac{f''(x_0)}{2}$.

4. ~~Let E~~ ~~let the~~ Assume a random partition on $[0,1]$: $[~~P_0, P_1~~ P_1, P_2, \cdots P_n]$, where ~~$P_0=0$~~ $P_1=0$, $P_n=1$.

$U(P,f)=~~P_0^2+P_1~~\ P_1^2+P_2^2(P_2-P_1)+\cdots+P_n^2(P_n-P_{n-1})$ ($\mathbb{Q}$ dense in $\mathbb{R}$)

$=P_1^2+P_2^2-P_2P_1+\cdots+P_n^2-P_nP_{n-1}$

$2U(P,f)=2P_1^2+2P_2^2-2P_1P_2+\cdots+2P_n^2-2P_nP_{n-1}$

$=P_1^2+(P_1^2+P_2^2-2P_1P_2)+(P_2^2+P_3^2-2P_2P_3)+\cdots+(P_{n-1}^2+P_n^2-2P_nP_{n-1})+P_n^2$

$\geq P_1^2+P_n^2=1$.

$U(P,f)\geq\frac{1}{2}\Rightarrow \overline{\int_a^b} f(x)dx=\inf_P U(P,f)=\frac{1}{2}$.

$\mathbb{R}\setminus\mathbb{Q}$ dense in $\mathbb{R}\Rightarrow$ similar reasoning, $\underline{\int_a^b} f(x)dx=\sup_P L(P,f)=0$.

We see upper and lower integrals ~~to be~~ not to agree. Hence f is not Riemann integrable.

5. ✓

Can use L'H.

a) ~~Yes~~ $\frac{\infty}{\infty}$ case. $\lim\limits_{x\to\infty} \frac{x-\sin x}{2x+\sin x} = \lim\limits_{x\to\infty} \frac{1-\cos x}{2+\cos x} \Rightarrow$ DNE.

Original limit: $\lim\limits_{x\to\infty} \frac{1-\sin x/x}{2+\sin x/x} = \cancel{\lim} \frac{1-0}{2+0} = \frac{1}{2}$.

b) $\lim\limits_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{1/x} = \lim\limits_{x\to 0^+} e^{\ln\left(\frac{\sin x}{x}\right)^{1/x}} = \lim\limits_{x\to 0^+} e^{\frac{1}{x}\ln\frac{\sin x}{x}} = e^{\lim\limits_{x\to 0^+} \frac{\ln\frac{\sin x}{x}}{x}}$ ✓ $\frac{0}{0}$ case. Apply L'H.

$\lim\limits_{x\to 0^+} \ln\frac{\sin x}{x} = \ln \lim\limits_{x\to 0^+} \cos x = \ln 1 = 0$.

$\lim\limits_{x\to 0^+} \frac{\ln\frac{\sin x}{x}}{x} = \lim\limits_{x\to 0^+} \frac{x\cos x-\sin x}{x\sin x} = \lim\limits_{x\to 0^+} \frac{-x\sin x}{x\cos x+\sin x} = \lim\limits_{x\to 0^+} -x\cos x \frac{1}{\sin x/x\cos x+1}$

$= \lim\limits_{x\to 0^+} \frac{-x}{1/\cos x \cdot \cancel{x/\sin x}\ \sin x/x + 1} = \lim\limits_{x\to 0^+} \frac{-x}{1\cdot 1+1} = \cancel{\lim} \frac{0}{2} = 0$.

Original limit: $e^0 = 1$.

c) $\frac{\infty}{\infty}$ case, apply L'H. ✓ ~~lim~~

~~To calculate original limit,~~ The original limit DNE.

If it exists, then $= \lim\limits_{x\to\infty} \frac{2x+\sin 2x+1}{2x+\sin 2x} \cdot \lim\limits_{x\to\infty} \frac{1}{(\sin x+3)^2} = \lim\limits_{x\to\infty} \frac{1}{(\sin x+3)^2} \Rightarrow$ this DNE, contradiction.

6. $f'(x_0) = \lim\limits_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$. This must exist.

~~$f''(x_0) = \lim\limits_{h\to 0} \frac{f'(x_0+h)-f'(x_0)}{h} = \lim\limits_{h\to 0}$~~

$\lim\limits_{h\to 0} \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} = \lim\limits_{h\to 0} \left(\frac{f(x_0+h)-f(x_0)}{h} - \frac{f(x_0)-f(x_0-h)}{h}\right)/h = \lim\limits_{h\to 0} \frac{f'(x_0)-f'(x_0-h)}{h} = f''(x_0)$.

7. $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$

let $x_0 = 0$. $f(x) = 1 + \frac{1}{2}x + (-\frac{1}{4})x^2/2 = 1 + \frac{x}{2} - \frac{x^2}{8}$.

Error term: $\exists \varepsilon \in (-\frac{1}{2}, \frac{1}{2}) \Rightarrow -\frac{1}{8}(1+\varepsilon)^{-\frac{3}{2}}x^2 = |-(4(1+\varepsilon))^{-\frac{3}{2}}x^2|$

$= \frac{1}{(4(1+\varepsilon))^{3/2}} x^2$. To maximize this, let $\varepsilon = -\frac{1}{2}$.

↳ becomes $\frac{\sqrt{2}}{4}x^2$, still less than $\frac{|x|^3}{2}$.