

Solutions to Problem Set 10

Math 425a, Fall 2021

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1. Note that $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x-0} = f'(0)$. Therefore defining $g(0) := f'(0)$ we see that $g \in C((-1, 1))$.

We need to show that $g \in C^1((-1, 1))$ that is it is differentiable on $(-1, 1)$ and that g' is continuous on $(-1, 1)$.

Away from $x = 0$, $g(x)$ is defined as $f(x)/x$ which a quotient of two differentiable functions hence continuously differentiable (by Lem. 9.10). At $x = 0$,

$$g'(0) = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - hf'(0)}{h^2} \stackrel{H}{=} \frac{f''(0)}{2}$$

where the last equality follows from applying de l'Hôpital rule twice and the fact that $f \in C^2(-1, 1)$. Thus

$$g'(x) = \begin{cases} \frac{xf'(x)-f(x)}{x^2} & \text{if } x \neq 0 \\ \frac{f''(0)}{2} & \text{if } x = 0 \end{cases}$$

Last thing to check is that $g'(x)$ is continuous. It is obviously continuous away from 0 being a quotient of continuous functions. To check continuity at $x = 0$ we have

$$\lim_{x \rightarrow 0} g'(x) = \lim_{x \rightarrow 0} \frac{xf'(x) - f(x)}{x^2} \stackrel{H}{=} \frac{f''(0)}{2} = g'(0),$$

as required, where again the second equality follows from an application of de l'Hôpital rule and the fact that $f''(x)$ is a continuous function.

2. Since f vanishes for infinitely many points in $[-1, 1]$ and $[-1, 1]$ is a compact set we have by the Bolzano-Weierstrass theorem (Thm. 4.9) that there exists a sequence $(a_k)_{k \geq 1}$ of such points converging to some $a \in [-1, 1]$. We can assume that (a_k) is a monotonic sequence (as there are infinitely many a_k 's on at least one side of a (say on the positive side), and then on that side we can pick a subsequence $(a_{k_n})_{n \geq 1}$ such that $a_{k_n} \in (a, a_{k_{n-1}})$ for all $n \geq 2$). WLOG assume that (a_k) is decreasing.

By continuity of f (or by PS7.10), $f(a) = 0$ too. Now for each a_k there exists, by Rolle's Theorem (Thm. 9.15), a point $b_k \in (a_{k+1}, a_k)$ such $f'(b_k) = 0$. By squeeze theorem (Cor. 5.9), since $a_{k+1} < b_k < a_k$, we have that $b_k \rightarrow a$ too. Then again since f' is continuous (as $f \in C^\infty(-2, 2)$ all its derivatives exist and are continuous), we get

that $f'(a) = 0$. Continuing in this way, we can iteratively prove that $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$.

Now let N be such that for all $n > N$ we have $|f^{(n)}(x)| \leq Cn!$ for all $x \in (-2, 2)$ as per the given condition. Taking a Taylor expansion around a of order n gives us

$$f(x) = \frac{f^{(n)}(\theta)(x-a)^n}{n!}$$

for some θ lying between a and x .

Using the given bound therefore we have that for all $n > N$, $|f(x)| \leq C|x-a|^n$. Now if x is such that $|x-a| < 1$ then taking limits as $n \rightarrow \infty$ we get that $f(x) = 0$ for all $x \in (a-1, a+1)$ which gives us the required open interval.

3. (a) That $f(x_0) = a_0$ can be verified by directly substituting $x = x_0$ in the given local expansion. Also,

$$\begin{aligned} f'(x_0) &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + o((x-x_0)^n)}{x-x_0} \\ &= a_1 \end{aligned}$$

- (b) One can verify via direct computation that

$$f'(x) = \begin{cases} 3x^2 \cos \frac{1}{x} + x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Thus $\lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x} = \lim_{x \rightarrow 0} \frac{3x^2 \cos \frac{1}{x} + x \sin \frac{1}{x}}{x} = \lim_{x \rightarrow 0} (3x \cos \frac{1}{x} + \sin \frac{1}{x})$ does not exist (as we have limit 1 for $x_n := 1/(\pi/2 + 2n\pi)$, and limit -1 for $y_n := 1/(3\pi/2 + 2n\pi)$, and so, by Heine's definition (Thm. 7.2), there is no limit).

However, $x^3 \cos \frac{1}{x} = o(x^2)$ (as $x \cos \frac{1}{x} \rightarrow 0$ as $x \rightarrow 0$), and so it is its own local expansion. In other words,

$$x^3 \cos \frac{1}{x} = 0 + 0 \cdot x + 0 \cdot x^2 + o(x^2),$$

which gives us the required local expansion.

4. Note that on any subinterval $[a, b] \subset [0, 1]$, we have that

$$\sup_{[a,b]} f = b, \quad \inf_{[a,b]} f = 0.$$

Indeed, the case of the infimum is obvious and, as for the supremum, we can find a sequence of rational numbers in $[a, b]$ that converge to b and since f is the identity

on rationals (and is bounded by b on $[a, b]$) we can use Ex. 3.11 to conclude that $\sup_{[a,b]} f = b$.

Now let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[0, 1]$. On any such partition $L(P, f) = 0$ and, as for $U(P, f)$, we can use the above discussion (with $[a, b] := [x_{k-1}, x_k]$) to get

$$U(P, f) = \sum_{k=1}^n x_k(x_k - x_{k-1}) = \sum_{k=1}^n x_k^2 - \underbrace{x_k x_{k-1}}_{\leq x_k^2/2 + x_{k-1}^2/2} \geq \sum_{k=1}^n \left(\frac{x_k^2}{2} - \frac{x_{k-1}^2}{2} \right) = \frac{x_n^2}{2} - \frac{x_0^2}{2} = \frac{1}{2}.$$

Thus $\int_0^1 f dx = 0$ and $\bar{\int}_0^1 f dx \geq 1/2$.

However, letting $P_n := \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ we have

$$U(P_n, f) = \sum_{k=1}^n \frac{k}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{k=1}^n k = \frac{n(n+1)}{2n^2} \rightarrow \frac{1}{2}$$

as $n \rightarrow \infty$. This shows (by Ex. 3.11) that $\bar{\int}_0^1 f dx = 1/2$.

In particular, since $\bar{\int}_0^1 f dx \neq \int_0^1 f dx$, we conclude that f is not Riemann integrable on $[0, 1]$.

5. (a) Applying de l'Hôpital rule (Thm. 9.20) once gives us

$$\lim_{x \rightarrow \infty} \frac{x - \sin x}{2x + \sin x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1 - \cos x}{2 + \cos x}$$

The right hand side limit obviously does not exist so de l'Hôpital is not useful here. However the limit can be evaluated as follows

$$\lim_{x \rightarrow \infty} \frac{x - \sin x}{2x + \sin x} = \lim_{x \rightarrow \infty} \frac{1 - \frac{\sin x}{x}}{2 + \frac{\sin x}{x}} = \frac{1}{2}$$

because $\frac{\sin x}{x}$ and $\frac{\cos x}{x}$ both converge to 0 as $x \rightarrow \infty$ by the squeeze theorem (Cor. 5.9).

- (b) Let us look at $\lim_{x \rightarrow 0^+} \frac{1}{x} \ln \left(\frac{\sin x}{x} \right)$. This is of the form $\frac{0}{0}$ so repeated application of de l'Hôpital rule gives us

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{1}{x} \ln \left(\frac{\sin x}{x} \right) &\stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{x \cos x - \sin x}{x \sin x} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{-x \sin x}{\sin x + x \cos x} \\ &\stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{-x \cos x - \sin x}{2 \cos x - x \sin x} \\ &= 0 \end{aligned}$$

Since $\left(\frac{\sin x}{x} \right)^{\frac{1}{x}} = e^{\frac{1}{x} \ln \left(\frac{\sin x}{x} \right)}$ we conclude that the limit in question is 1.

- (c) Note that de l'Hôpital rule cannot be used as the assumption that $g'(x) \neq 0$ is not satisfied. We have that

$$\frac{2x + \sin 2x + 1}{(2x + \sin 2x)(\sin x + 3)^2} = \frac{2 + \frac{\sin 2x}{x} + \frac{1}{x}}{2 + \frac{\sin 2x}{x}} \cdot \frac{1}{(\sin x + 3)^2}.$$

The limit of the first factor is 1, but the second factor converges to different values for $x_n := \pi/2 + 2\pi n$ and $y_n := 2\pi n$. Thus the limit does not exist.

6. Note that since $f''(x_0)$ exists, so does $f'(x_0)$, and so f is continuous at x_0 (by Lem. 9.9). Thus the numerator of the expression inside the limit converges to 0 as $h \rightarrow 0$. Thus we can use de l'Hôpital rule to get

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} &\stackrel{H}{=} \lim_{h \rightarrow 0} \frac{f'(x_0 + h) - f'(x_0 - h)}{2h} \\ &= \lim_{h \rightarrow 0} \left(\frac{f'(x_0 + h) - f'(x_0)}{2h} + \frac{f'(x_0) - f'(x_0 - h)}{2h} \right) \\ &= \frac{f''(x_0)}{2} + \frac{f''(x_0)}{2} = f''(x_0). \end{aligned}$$

7. Note that $f(x) = \sqrt{x+1}$ is C^∞ in $(-\frac{1}{2}, \frac{1}{2})$. Therefore we can use Taylor's formula with Lagrange form as remainder (Thm. 10.4). Taking an expansion up to order 3 around 0 gives us,

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{3!8}(1+\alpha)^{-\frac{5}{2}}x^3$$

for some α between 0 and x .

Bounding the error term, we have,

$$\left| \frac{3}{3!8}(1+\alpha)^{-\frac{5}{2}}x^3 \right| < \frac{3|x|^3}{48} < \frac{1}{2}|x|^3$$

as required.

8. Applying Taylor's formula with remainder in Lagrange form to $f(x) = (1+x)^\alpha$ we get,

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)h^{\alpha-2}}{2}x^2$$

for some $h \in (0, x)$.

Now it is enough to observe that

$$\frac{\alpha(\alpha-1)h^{\alpha-2}}{2}x^2 > 0 \text{ if } \alpha > 1 \text{ or } \alpha < 0$$

and

$$\frac{\alpha(\alpha-1)h^{\alpha-2}}{2}x^2 < 0 \text{ if } \alpha \in (0, 1)$$

Note here that the inequalities are strict because $x \neq 0$ by assumption.

9. Following the hint, we write

$$f(x+1) = f(x) + f'(x) + \frac{1}{2}f''(\psi)$$

for some $\psi \in (x, x+1)$. Rearranging the above equation we get,

$$\begin{aligned} xf'(x) &= xf(x+1) - xf(x) - \frac{x}{2}f''(\psi) \\ &= \frac{x}{x+1}(x+1)f(x+1) - xf(x) - \frac{1}{2}\frac{x}{\psi}\psi f''(\psi) \end{aligned}$$

Now let us look at the individual terms on the right hand side. Since we are interested in the limit as x tends to $+\infty$ we can assume $x > 0$. Then $0 < \frac{x}{x+1} < 1$ and by assumption $(x+1)f(x+1) \rightarrow 0$. Therefore $\frac{x}{x+1}(x+1)f(x+1) \rightarrow 0$ as $x \rightarrow \infty$. $xf(x)$ also tends to 0 as $x \rightarrow \infty$ from hypothesis. For the last term note that $\psi \in (x, x+1)$ so $\psi \rightarrow \infty$ as $x \rightarrow \infty$. Then $\lim_{x \rightarrow \infty} \psi f''(\psi) = \lim_{\psi \rightarrow \infty} \psi f''(\psi) = 0$ from given conditions. Also $0 < \frac{x}{\psi} < 1$ which implies $\frac{1}{2}\frac{x}{\psi}\psi f''(\psi) \rightarrow 0$ too.

Since all the terms in the RHS converge to 0 as $x \rightarrow \infty$ we have $\lim_{x \rightarrow \infty} xf'(x) = 0$, as desired.

10. Recall that local expansions are valid in a neighborhood around x_0 . So we have one neighborhood where the expansion of f is valid and one where the expansion of g is valid. But then, on the smaller of the two neighborhoods both the expansions are valid. On such neighbourhood we can write,

$$\begin{aligned} f(x)g(x) &= \left(a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + o((x - x_0)^2) \right) \\ &\quad \left(b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + o((x - x_0)^2) \right) \end{aligned}$$

Multiplying out and putting all higher powers than 2 in to the order term we then have

$$\begin{aligned} f(x)g(x) &= a_0b_0 + (a_0b_1 + a_1b_0)(x - x_0) + (a_0b_2 + a_1b_1 + a_2b_0)(x - x_0)^2 + o((x - x_0)^2), \end{aligned}$$

since $a(x - x_0)^k o((x - x_0)^2)$ is also $o((x - x_0)^2)$ for any $a \in \mathbb{R}$, $k \geq 0$. This gives us the required local expansion along with the coefficients.

11. That P is a partition is obvious. Next note that $f(x) = x^2$ is an increasing function on $[0, 1]$ which gives us that the supremum on any interval occurs at the right end point and the infimum occurs at the left. Using that we have (directly from Def. 10.10),

$$\begin{aligned} L(P, f, \alpha) &= \sum_{k=1}^{k=4} \frac{(k-1)^2}{16} \left(\alpha \left(\frac{k}{4} \right) - \alpha \left(\frac{k-1}{4} \right) \right) \\ &= \sum_{k=1}^{k=4} \frac{(k-1)^2}{16} \frac{1}{4} \\ &= \frac{1}{64}(0 + 1 + 4 + 9) = \frac{14}{64}. \end{aligned}$$

Similarly,

$$\begin{aligned} U(P, f, \alpha) &= \sum_{k=1}^{k=4} \frac{k^2}{16} \left(\alpha \left(\frac{k}{4} \right) - \alpha \left(\frac{k-1}{4} \right) \right) \\ &= \sum_{k=1}^{k=4} \frac{k^2}{16} \frac{1}{4} \\ &= \frac{1}{64} (1 + 4 + 9 + 16) = \frac{15}{32}. \end{aligned}$$