Math 425A Problem Sheet 11 (due 9am on Monday, 15th Nov)

Essential problems

1. (3 pts) Suppose that $f, \alpha : [a,b] \to \mathbb{R}$ are such that there exists $x_0 \in (a,b)$ such that α is discontinuous at x_0 and f is discontinuous at x_0 from both sides (i.e. $\lim_{x\to x_0^+} f(x) \neq f(x_0)$ and $\lim_{x\to x_0^-} f(x) \neq f(x_0)$). Show that $f \notin \mathcal{R}(\alpha)$. *(Comment: note that if only f is discontinuous at c, then $f \in \mathcal{R}(\alpha)$ by Thm. 11.2 (see also question 6 below), and that if f is continuous, but α is not, then also $f \in \mathcal{R}(\alpha)$ by Thm. 10.17.)*

2. (2 pts) Suppose that $f \in \mathcal{R}$ on $[a,b]$. Show that redefining the values of f at finitely many points does not influence the integrability of f and the value of $\int_a^b f\,\mathrm{d}x$. Why does it not contradict question 1? Show that this claim is false if one replaces "finitely" by "infinitely" (i.e. find a counterexample).

3. (3 pts) Let $c \in \mathbb{R}$ and

$$f(x) := \begin{cases} \sin\frac{1}{x} & x \neq 0, \\ c & x = 0. \end{cases}$$

For which values of c does f have an antiderivative (i.e. $F\colon \mathbb{R} \to \mathbb{R}$ such that $F' = f$)? *(Comment: note that this is a part of the "topologist's curve" that we have seen in Ex. 0.1 and Ex. 9.12.2; also hint: you might want to use integration by parts (Cor. 11.12) to find that $\int_a^b \sin(1/x)\mathrm{d}x = \left[x^2\cos(1/x)\right]_a^b - 2\int_a^b x\cos(1/x)\mathrm{d}x$ for any interval $[a,b]$ not containing 0.)*

4. (2 pts) Show that if $f \in C([a,b])$ and $g \in C^1([a,b])$ is monotonic then there exists $\xi \in (a,b)$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^\xi f(x)\mathrm{d}x + g(b)\int_\xi^b f(x)\mathrm{d}x.$$

(Comment: this is the so-called Second Mean Value Theorem for integrals; also hint: let $F(x) := \int_a^x f(t)\,\mathrm{d}t$, use Thm. 11.10 and then integration by parts (Cor. 11.12) and the "first" Mean Value Theorem for Integrals (Lem. 11.18).) Use this fact to find

$$\lim_{n\to\infty} \int_a^b \frac{\sin(nx)}{x}\,\mathrm{d}x,$$

where $0 < a < b$. Why can't one use the "first" Mean Value Theorem for integrals (i.e. Lem. 11.18) to calculate this limit?

Additional problems

5. (1 pt) Suppose that $f, g \in \mathcal{R}(\alpha)$ on an interval $[a,b]$. Show that $\max(f,g), \min(f,g) \in \mathcal{R}(\alpha)$.

6. (1 pt) Let $c \in (0,1)$ and $f\colon [0,1] \to \mathbb{R}$ be such that $f(c) := 1$ and $f(x) := 0$ for $x \neq c$. Let $\alpha\colon [0,1] \to \mathbb{R}$ be an increasing function that is continuous at c. Show that $f \in \mathcal{R}(\alpha)$ on $[0,1]$ and $\int_0^1 f\mathrm{d}\alpha = 0$, without using question 2 or Thm. 11.2.

7. (1 pt) Calculate each the following integrals, if they exist.

(a) $\int_0^2 \cos(x^2)6x\,\mathrm{d}x$,
(b) $\int_1^2 (\ln x)/x^2\,\mathrm{d}x$,

(c) $\int_{-1}^{1} x^3 \,\mathrm{d}\alpha(x)$, where

$$\alpha(x) := \begin{cases} 2 & x \in [-1,0], \\ x^2+3 & x \in (0,1]. \end{cases}$$

8. (1 pt) Show that if $f \in C([a,b])$ is a nonnegative function such that

$$\int_a^b f \,\mathrm{d}x = 0$$

then $f = 0$. Deduce that if $f \in C([a,b])$ is such that

$$\int_a^b fg \,\mathrm{d}x = 0$$

for all $g \in C([a,b])$ then $f = 0$. *(Comment: This can be thought of as a 1-dimensional version of the Fundamental Theorem of Calculus of Variations.)*

9. (1 pt) Let $f \in C(\mathbb{R})$ be periodic with period $c > 0$. Show that

$$\int_0^c f(x) \,\mathrm{d}x = \int_z^{z+c} f(x) \,\mathrm{d}x$$

for every $z \in \mathbb{R}$.

10. (1 pt) Find $\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x} \mathrm{d}x$. *(Hint: Use the Mean Value Theorem for Integrals (Lem. 11.18) with $g(x) := x^n$.)*

11. (1 pt) Show that

$$\int_0^\pi x f(\sin x) \,\mathrm{d}x = \frac{\pi}{2} \int_0^\pi f(\sin x) \,\mathrm{d}x$$

for every $f \in C([0,1])$. *(Hint: Use Lem. 11.16 with $\phi(x) := \pi - x$.)*. Deduce that

$$\int_0^\pi \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} \mathrm{d}x = \frac{\pi^2}{4}.$$

(Hint: Observe that $|\sin x| = |\cos(x - \pi/2)|$ and $|\cos x| = |\sin(x - \pi/2)|$.)

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