MATH 425a PS 11

1. Since both α and f discontinuous at x_0, consider $P := [x_0, \cdots, x_{i-1}, m, x_i, \cdots x_n]$, s.t. $a = \cancel{x_0}^{\alpha(x_0)}, b = \cancel{x_n}^{\alpha(x_n)}$

let $x_i - m = \delta_i$. $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists y_1 \in (m, x_i)$ s.t. $|f(y_1) - f(m)| \geq \varepsilon$ ~~if $|y_1 - m| \leq \delta$~~.

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists y_2 \in (m, x_i)$ s.t. $|\alpha(y_2) - \alpha(m)| \geq \varepsilon$.

$U(P, f, \alpha) - L(P, f, \alpha) \geq [\underset{x \in [m, x_i]}{M f(x)} - \underset{x \in [m, x_i]}{m f(x)}][\alpha(x_i) - \alpha(m)]$. (leave other non-negative terms).

Notice that the first term must be $\geq \varepsilon$. $\alpha(x_i) - \alpha(m) \geq \alpha(y_2) - \alpha(m) \geq \varepsilon$. aside

as $f(y_1) - f(m) \geq \varepsilon$ as ~~covers the whole interval~~ $\alpha(x_i) \geq \alpha(y_2)$ (WLOG).

Hence $U(P, f, \alpha) - L(P, f, \alpha) \geq \varepsilon^2$. Since ε is arbitrary, $f \notin R(\alpha)$.

2. Consider partition $P := [x_0, x_1, \cdots x_n]$, where $x_0 = a$, $x_n = b$.

Since $f \in R$, $U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i < \varepsilon \quad \forall \varepsilon > 0$.

Suppose there are m values of $f(x)$ redefined, then at most $m(M_i - m_i)$ values will change. WLOG assume they all increase.

We denote $M_i - m_i = \Delta f_i$. Then ~~pick for~~ for each Δf_i term (m terms at most),

pick n_i so large s.t. $\Delta x_i < \frac{x_n - x_0}{n_i}$, $\Delta f_i \cdot \frac{x_n - x_0}{n_i} < \varepsilon$, $\forall \varepsilon$.

$n_i > \frac{\Delta f_i (x_n - x_0)}{\varepsilon}$

Then $\sum_{i=1}^{n} \Delta f_i \Delta x_i < m\varepsilon$. Since ε is arbitrary, $\sum_{i=1}^{n} \Delta f_i \Delta x_i < \varepsilon$.

After the change $U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i + \sum_{i=1}^{n} \Delta f_i \Delta x_i < (1+m)\varepsilon \quad \forall \varepsilon > 0$.

Hence f is still Riemann integrable. $|U(P, f) - \int_a^b f(x)dx| < (1+m)\varepsilon$ ↳ arbitrarily small.

Thus $\int_a^b f(x)dx$ does not change.

Counterexample: this is easy to find. $\forall x \in [a, b]$ (infinitely many), let $g := 2f(x)$. Then $\int_a^b g(x)dx \neq \int_a^b f(x)dx$.

4. By integration by parts we have $\int_a^b f(x)g(x)dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx$

By MVT we have $\int_a^b f(x)g(x)dx = f(b)g(b) - F(a)g(a) - F(c)[g(b) - g(a)]$, $c \in [a, b]$

We know $F(x)$ is continuous and WLOG assume $g'(x)$ is positive since $g(x)$ monotonic.

$\int_a^b f(x)g(x)dx = g(a)(F(c) - F(a)) + g(b)(F(b) - F(c))$

$= g(a)\int_a^{\xi} f(x)dx + g(b)\int_{\xi}^{b} f(x)dx$ since c is a dummy variable.

$\int_a^b \frac{\sin(nx)}{x} dx = \frac{1}{a}\int_a^{\xi} \sin(nx)dx + \frac{1}{b}\int_{\xi}^{b} \sin(nx) dx$, $\xi \in (a, b)$.

$= \frac{1}{an}[-\cos(nx)]_a^{\xi} + \frac{1}{bn}[-\cos(nx)]_{\xi}^{b}$

$= \frac{1}{an}(-\cos n\xi) + \frac{1}{an}\cos an - \frac{1}{bn}\cos bn + \frac{1}{bn}\cos n\xi \Rightarrow \lim_{n \to \infty} \int_a^b \frac{\sin(nx)}{x} dx = 0$.

$= \cancel{\left(\frac{1}{bn} - \frac{1}{an}\right) - \frac{1}{n}\left(\frac{1}{b} - \frac{1}{a}\right)\cos n\xi}$ Cannot use 11-18: $\sin(nx)$ not necessarily ≥ 0.

3. $c = 0$. To make $F'(x) = f(x)$, observe $F(x) = x^2 \cos\frac{1}{x} - 2\int_0^x t\cos\frac{1}{t}dt$.

~~$F'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\sin\frac{1}{h} - 0}{h} = 0 = f(0)$~~. $\lim_{h \to 0} \frac{F(h) - F(0)}{h} = \lim_{h \to 0} \frac{F(h)}{h} = 0 = f(0)$. $\Rightarrow f$ has antiderivative when $c = 0$.

5. By definition, $f, g \in R(\alpha)$ implies both f, g are bounded. Also, according to Lebesgue's criterion for Riemann integrability, f and g must be continuous ~~alwa~~ almost everywhere. It's trivial that if $f, g \in R(\alpha)$, then $\max(f,g)$, $\min(f,g)$ must be bounded. Since there are ~~finitely~~ at most countably many discontinuous points on both f and g, $\max(f,g)$ and $\min(f,g)$ have ~~finitely~~ at most countably many discontinuous points, hence $\max(f,g), \min(f,g) \in R(\alpha)$.

↳ finite union of countable sets

6. We pick a partition $P := [x_0, x_1, \cdots x_n]$ s.t. $x_0 = 0$, $x_n = 1$, let ~~$c \in [x_{i-1}, x_i]$, $c \in (0,1)$, $c \notin \{x_i\}_{i\in[0,n]}$~~ → let $c \in (x_{i-1}, x_i)$, $i \in [1, n]$.

$U(P,f,\alpha) - L(P,f,\alpha) = 0 + (\alpha(x_i) - \alpha(x_{i-1})) \cdot 1 - 0$
$= \alpha(x_i) - \alpha(x_{i-1})$.

We pick $n \in \mathbb{N}$ so large s.t. $\alpha(x_i) - \alpha(x_{i-1}) < \frac{\alpha(1)-\alpha(0)}{n}$.

Then $\forall \varepsilon > 0$, $\exists n > \frac{\alpha(1)-\alpha(0)}{\varepsilon}$ s.t. $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

Thus $f \in R(\alpha)$ on $[0,1]$. $\because f \in R(\alpha)$, $\int_0^1 f d\alpha = \int_0^1 f(x) dx = 0$.

7. a) $\int_0^2 \cos(x^2) 6x \, dx = 3[\sin(x^2)]_0^2 = 3\sin 4$.

b) $\int_1^2 \frac{\ln x}{x^2} dx = [\ln x \cdot (-\frac{1}{x})]_1^2 - \int_1^2 (-\frac{1}{x^2}) dx = -\frac{1}{2}\ln 2 + [-\frac{1}{x}]_1^2 = -\frac{1}{2}\ln 2 + \frac{1}{2}$.

c) $\int_{-1}^1 x^3 d\alpha(x) = 0 + \int_0^1 x^3 \cdot (2x) dx = [\frac{2x^5}{5}]_0^1 = \frac{2}{5}$.

9. Assuming Riemann integrability, let $F(x) = \int_a^x f(t) dt$, $a \le \min(0, z)$.

Since f is periodic, $f(0) = f(c)$; $f(z) = f(z+c)$.

~~Thus~~ By FTC $F(c) - F(0) = F(z+c) - F(z)$. $\Rightarrow \int_0^c f(x) dx = \int_z^{z+c} f(x) dx$ $\forall z \in \mathbb{R}$.

10. $\forall x \in [0,1]$, let $g(x) := x^n \ge 0$. By MVT for integrals, $\exists \xi \in (a,b)$.

s.t. $\int_0^1 \frac{x^n}{1+x} dx = \frac{1}{1+\xi} \int_0^1 x^n dx = \frac{1}{1+\xi} [\frac{x^{n+1}}{n+1}]_0^1 = \frac{1}{n+1}$. $\Rightarrow \lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x} dx = 0$.

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