Homework Assignment Solution 11

November 19, 2021

1. (Let us denote x_0, given by the question, by c) Since α is discontinuous at c either

$$\lim_{x\to c^+} \alpha(x) > \alpha(c) \quad \text{or} \quad \lim_{x\to c^-} \alpha(x) < \alpha(c),$$

due to Thm. 9.6. Let's assume that the former is the case (the other case is analogous). Then there exists $\sigma > 0$ such that $\alpha(c) + \sigma \leq \alpha(x)$ for all $x > c$ (recall α is nondecreasing).

Let $P = (x_k)_{k=0}^n$ be any partition, and let $k \in \{0, 1, \dots, n\}$ be such that $c \in [x_{k-1}, x_k)$ (note that such k exists as $c \in (a, b)$). Then $\alpha(x_k)-\alpha(x_{k-1}) \geq \alpha(x_k)-\alpha(c) \geq \sigma$.

Furthermore, since f is discontinuous at c from the right, there exists $\delta > 0$ and $x \in (c, x_k)$ such that $|f(x) - f(c)| \geq \delta$. In particular

$$\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \geq \delta,$$

and so

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right) (\alpha(x_i) - \alpha(x_{i-1})) \\ &\geq \left(\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \right) (\alpha(x_k) - \alpha(x_{k-1})) \geq \delta\sigma. \end{aligned}$$

Thus the condition (10.2) cannot be satisfied for $\varepsilon < \sigma\delta$, and so $f \notin \mathcal{R}(\alpha)$ (by Thm. 10.15).

2. Let g be the function that we obtain after redefining f at points $x_1, \dots, x_n$, by adding $c_1, \dots, c_n$, respectively. Namely

$$g(x) = f(x) + c_1 I_{\{x_1\}}(x) + \dots + c_n I_{\{x_n\}}(x),$$

where $I_A(x) := 1$ if $x \in A$ and $I_A(x) := 0$ if $x \notin A$ denotes the indicator function (note that we have used the same notation of the indicator function in Ex. 13.10). Since $I_{\{x_i\}} \in \mathcal{R}$ on $[a, b]$ with $\int_a^b I_{\{x_i\}}(x)\mathrm{d}x = 0$ for every $i \in \{1, \dots, n\}$

(by question 6, which works also in the case even if $c \in [0,1]$, see below), and so $g \in \mathcal{R}$ on $[a,b]$ with $\int_a^b g(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$ (by Lem. 11.4(a)).

This does not contradict question one, as in the case of Riemann integral $\alpha(x) = x$ is continuous.

PS10.4 shows that the claim is false if one redefines a function at countably many points.

3. Let us denote by $f_c(x)$ (with $c \in \mathbb{R}$) the function

$$f_c(x) := \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ c & \text{if } x = 0. \end{cases}$$

We first show that for $f_0(x)$, there exists an antiderivative. If we denote by $g_n(x)$ (with a positive integer n) the continuous function

$$g_n(x) := \begin{cases} x^n \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

then for every n, $g_n(x)$ is Riemann integrable on every closed interval. Using the idea given by the hint, we see that

$$F(x) := g_2(x) - 2\int_0^x g_1(x)\mathrm{d}x$$

is a differentiable function (even at 0), with the derivative equal to $\sin(1/x)$ for every $x \neq 0$, and at $x = 0$,

$$F'(0) = \lim_{x\to 0} \frac{F(x) - F(0)}{x} = \lim_{x\to 0} \frac{F(x)}{x} = \lim_{x\to 0}\left(g_1(x) - \frac{2}{x}\int_0^x g_1(x)dx\right) = 0,$$

because by the MVT for integrals, there exists some $c \in [0,x]$ such that

$$\frac{1}{x}\int_0^x g_1(x)dx = g_1(c),$$

which then goes to zero as $x \to 0$ as $g_1(x)$ is continuous. Hence for $f_0(x)$, $F(x)$ is an antiderivative.

Assume now that c is a value for which $f_c(x)$ has an antiderivative. Observe that

$$(f_c - f_0)(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ c & \text{if } x = 0. \end{cases}$$

If G were an antiderivative of f_c, then $G - F$ is an antiderivative of $f_c - f_0$. By Darboux's theorem (Thm 9.19), $G' - F'$ must have the intermediate value property. However, $G' - F' = f_c - f_0$ takes only values 0 or c, which contradicts the intermediate value property. Therefore the only value of c for which $f_c(x)$ can have an antiderivative is $c = 0$.

4. Without loss of generality assume that g is nondecreasing (the other case is analogous). By Cor. 9.18 $g' \geq 0$. Following the hint, let $F(x) := \int_a^x f(t)dt$. By integration by parts,

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= \int_a^b F'(x)g(x)dx \\
&= F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)dx \\
&= F(b)g(b) - \int_a^b F(x)g'(x)dx \\
&= g(b)\int_a^b f(x)dx - \int_a^b F(x)g'(x)dx.
\end{aligned}$$

By Lem. 11.18, there exists some $\xi \in (a,b)$ such that

$$\int_a^b F(x)g'(x)dx = F(\xi)\int_a^b g'(x)dx = (g(b)-g(a))\int_a^\xi f(x)dx.$$

Thus

$$\begin{aligned}
\int_a^b f(x)g(x)dx &= (g(a)-g(b))\int_a^\xi f(x)dx + g(b)\int_a^b f(x)dx \\
&= (g(a)-g(b))\int_a^\xi f(x)dx + g(b)\int_a^\xi f(x)dx + g(b)\int_\xi^b f(x)dx \\
&= g(a)\int_a^\xi f(x)dx + g(b)\int_\xi^b f(x)dx.
\end{aligned}$$

Using this, we can compute $\lim_{n\to\infty}\int_a^b \sin(nx)/x dx$ as follows: Let $f(x) = \sin(nx)$ and $g(x) = 1/x$. Then by the FTC,

$$\begin{aligned}
\left|\int_a^b \frac{\sin(nx)}{x}dx\right| &= \left|\frac{1}{a}\int_a^\xi \sin(nx)dx + \frac{1}{b}\int_\xi^b \sin(nx)dx\right| \\
&= \left|\frac{1}{a}\frac{\cos(an)-\cos(\xi n)}{n} + \frac{1}{b}\frac{\cos(\xi n)-\cos(bn)}{n}\right| \\
&\leq \frac{2}{an} + \frac{2}{bn} \to 0
\end{aligned}$$

as $n \to \infty$. If we used the first MVT (Lem. 11.18), then we would get $\sin(\xi n)(\ln b - \ln a)$, from where we are unable to calculate the limit in n. Note that we cannot use Lem. 11.18 with $g(x) := \sin(nx)$, as it is not a nonnegative function.

5. Note from PS1.4, $\max\{f,g\} = (|f-g|+f+g)/2$ and similarly for $\min\{f,g\}$. Since $x \mapsto |x|$ is continuous (by PS8.7(b)), by Lem. 11.4(a), both $\max\{f,g\}$ and $\min\{f,g\}$ are Riemann-Stieltjes integrable.

6. Let $\epsilon > 0$ be given. Let δ be such that for every x satisfying $|x-c| \le \delta$, $|\alpha(x)-\alpha(c)| \le \epsilon/2$. Let $P := (0, c-\delta/2, c+\delta/2, 1)$. Then

$$\begin{aligned} U(P,f,\alpha)-L(P,f,\alpha) &= \alpha(c+\delta/2)-\alpha(c-\delta/2) \\ &= \alpha(c+\delta/2)-\alpha(c)+\alpha(c)-\alpha(c-\delta/2) \\ &\le \epsilon. \end{aligned}$$

Hence by Thm. 10.15, $f \in \mathcal{R}(\alpha)$, and since $L(Q,f)=0$ for any partition Q we get (by taking $\sup_Q$) $\underline{\int_0^1} f\mathrm{d}x = 0$, and so $\int_0^1 f\mathrm{d}x = 0$.

Note that the solution remains valid even if $c=0$ or $c=1$ (e.g. for the former case take $P := (0, \delta/2, 1)$).

7. (a) Observe that $(3\sin(x^2))' = 6x\cos(x^2)$. Hence by the FTC,

$$\int_0^2 \cos(x^2)6x\,dx = 3\sin 4.$$

(b) Observe that

$$\frac{d}{dx}\left(-\frac{1}{x}-\frac{\ln x}{x}\right) = \frac{\ln x}{x^2}.$$

Hence by the FTC,

$$\int_1^2 \frac{\ln x}{x^2}dx = \left(-\frac{1}{2}-\frac{\ln 2}{2}\right)-\left(-\frac{1}{1}-\frac{\ln 1}{1}\right) = \frac{1}{2}-\frac{\ln 2}{2}.$$

(c) Note that $f(x) := x^3$ belongs to $\mathcal{R}(\alpha)$ on $[-1,1]$ by Thm. 10.17. Observe that since α is constant on $[-1,0]$, $\alpha(x)-\alpha(y) = 0$ for every $x,y \in [-1,0]$. Thus (by Lem. 11.4(c))

$$\int_{-1}^1 x^3 d\alpha(x) = \int_{-1}^0 x^3 d\alpha(x) + \int_0^1 x^3 d\alpha(x) = \int_0^1 x^3 d\alpha(x).$$

Since α is increasing on $[0,1]$ and since $\alpha'(x) = 2x \in \mathcal{R}$, by Thm. 11.8, we have

$$\int_0^1 x^3 d\alpha(x) = \int_0^1 x^3(2x)dx = \int_0^1 2x^4 dx = \frac{2}{5}.$$

8. Suppose for contradiction that $f(c) > 0$ for some $c \in [a, b]$. Let $\epsilon := f(c)/2$. By continuity, there exists some δ such that for every x satisfying $|x - c| < \delta$, $f(x) - f(c) \geq -\epsilon = -f(c)/2$, i.e. $f(x) \geq f(c)/2$. Hence (since f is nonnegative)

$$0 = \int_a^b f(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^b f(x)dx$$
$$\geq \int_{c-\delta}^{c+\delta} f(x)dx \geq \frac{\delta f(c)}{2},$$

and this is a contradiction.

To show the second assertion, let $g = f$. Then f^2 is a nonnegative continuous function, hence by the previous assertion, $f^2 = 0 \iff f = 0$.

9. First note that we can assume that $z \in [0, c)$, since otherwise we have $z \in [kc, (k+1)c)$ and so letting $y := x - kc$ (i.e. letting $\alpha(x) := x$, $\varphi(y) := y + kc$, $A := z - kc$, $B := (z - kc) + c$, $a := z$, $b := z + c$ in Thm. 11.14) we have

$$\int_z^{z+c} f(x)dx = \int_{z-kc}^{(z-kc)+c} f(y+kc)dy = \int_{z-kc}^{(z-kc)+c} f(y)dy,$$

using periodicity.

We have (by Lem. 11.4(c))

$$\int_z^{z+c} f(x)dx = \int_z^c f(x)dx + \int_c^{z+c} f(x)dx.$$

Consider Thm. 11.14. Let $\alpha(x) = x$, $\varphi(x) = x + c$. Then $\varphi : [0, z] \to [c, z + c]$, $\beta(x) = \varphi(x)$, and $f(\varphi(x)) = f(x + c) = f(x)$, and so

$$\int_c^{z+c} f(x)dx = \int_c^{z+c} f(x+c)\mathrm{d}\alpha(x) \overset{Thm. 11.14}{=} \int_0^z f(\varphi(x))d\varphi(x) = \int_0^z f(x)dx.$$

Hence

$$\int_z^{z+c} f(x)dx = \int_z^c f(x)dx + \int_c^{z+c} f(x)dx = \int_z^c f(x)dx + \int_0^z f(x)dx = \int_0^c f(x)dx.$$

10. Let $g(x) := x^n$. This is a nonnegative function on $[0, 1]$, so by the MVT for integrals (Lem. 11.18), there exists some $\xi \in (0, 1)$ such that

$$\left|\int_0^1 \frac{x^n}{1+x}dx\right| = \left|\frac{1}{1+\xi}\int_0^1 x^n dx\right| \leq \frac{1}{1+n}$$

Hence as $n \to \infty$, the above integral tends to zero.

11. By Lem. 11.16 with $\phi(x) := \pi - x$, $\phi \colon [0,\pi] \to [0,\pi]$,

$$\begin{aligned}\int_0^\pi x f(\sin(x))dx &= \int_0^\pi (\pi - x) f(\sin x)dx \\ &= \pi \int_0^\pi f(\sin x)dx - \int_0^\pi x f(\sin x)dx.\end{aligned}$$

This gives the desired equality

$$\int_0^\pi x f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx.$$

Since $\cos^2 x = 1 - \sin^2 x$, observe that if we let

$$f(x) := \frac{x^8}{x^8 + (1-x^2)^4},$$

then $f(\sin x) + f(-\cos x) = 1$, and

$$\begin{aligned}\int_0^\pi \frac{x\sin^8 x}{\sin^8 x + \cos^8 x}dx &= \frac{\pi}{2}\int_0^\pi f(\sin x)dx \\ &= \frac{\pi}{2}\int_0^{\pi/2} f(\sin x)dx + \frac{\pi}{2}\int_{\pi/2}^{\pi} f(\sin(x))dx \\ &= \frac{\pi}{2}\int_0^{\pi/2} f(\sin x)dx + \frac{\pi}{2}\int_0^{\pi/2} f(\sin(x+\pi/2))dx \\ &= \frac{\pi}{2}\int_0^{\pi/2} f(\sin x)dx + \frac{\pi}{2}\int_0^{\pi/2} f(-\cos(x))dx \\ &= \frac{\pi}{2}\int_0^{\pi/2} f(\sin x) + f(-\cos(x))dx \\ &= \frac{\pi^2}{4}.\end{aligned}$$