Math 425A Problem Sheet 12 (due 9am on Monday, 22nd Nov)

Essential problems

1. (3 pts) Suppose that $f : (0,1) \to \mathbb{R}$ is nondecreasing and such that the improper integral $\int_0^1 f\,\mathrm{d}x$ converges. Show that

$$\lim_{n\to\infty} \frac{f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n-1}{n}\right)}{n} = \int_0^1 f\,\mathrm{d}x.$$

Use this fact to find

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}.$$

2. (2 pts) Suppose that $f\colon (0,\infty) \to (0,\infty)$ is differentiable and such that f' decreases to 0. Show that either both series

$$\sum_{n\geq 1} f'(n) \qquad \text{and} \qquad \sum_{n\geq 1} \frac{f'(n)}{f(n)}$$

converge or both diverge.

3. (2 pts) Let $(a_n)_{n\geq 1} \subset (0,\infty)$ be such that $\sum_{n\geq 1} a_n$ diverges. Let $S_n \coloneqq \sum_{k=1}^n a_k$. Show that

$$\sum_{n\geq 1} \frac{a_{n+1}}{S_n \log S_n} \quad \text{diverges, but} \quad \sum_{n\geq 1} \frac{a_n}{S_n (\log S_n)^2} \quad \text{converges.}$$

4. (3 pts) Let (X,d) be a metric space. Suppose that $f_n, g_n, g, f\colon X \to \mathbb{R}$ are such that that $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$ as $n \to \infty$.

(a) (2 pts) Show that $f_n + g_n \rightrightarrows f + g$, and that it is not true in general that $f_n g_n \rightrightarrows fg$ (find a counterexample),

(b) (0.5 pt) Show that the last claim is true if one replaces "$\rightrightarrows$" by the pointwise limit,

(c) (0.5 pt) Show that if, additionally, f, g are bounded, then $f_n g_n \rightrightarrows fg$.

Additional problems

5. (1 pt) Let $f \in C^{(n)}(\mathbb{R})$ be such that $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$. Show that there exists a constant $C_n > 0$, dependent only on n (and not on f), such that

$$\int_0^1 |f(x)|\mathrm{d}x \leq C_n \int_0^1 |f^{(n)}(x)|\mathrm{d}x.$$

(Comment: This is a simple form of an n-th order Poincaré inequality, which says, roughly speaking, that the size the highest derivative bounds the size of all lower derivatives. It is one of the fundamental inequalities in the area of differential equations.)

6. (1 pt) Let $-\infty \leq a < b \leq \infty$. Show that the claims of Lem. 11.4 remain valid for an improper integral $\int_a^b f\mathrm{d}x$; i.e. assume that $f, g \in \mathcal{R}$ on $[\alpha,\beta]$ for every $\alpha,\beta \in (a,b)$, $\alpha < \beta$, are such that $\int_a^b f\,\mathrm{d}x$, $\int_a^b g\,\mathrm{d}x$ both converge, and show that

(a) $\int_a^b (f + cg)\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x + c\int_a^b g\,\mathrm{d}x$,

(b) If $f \leq g$ then $\int_a^b f\,\mathrm{d}x \leq \int_a^b g\,\mathrm{d}x$,

(c) $\int_a^b f\,\mathrm{d}x = \int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x$ for every $c \in (a,b)$,

(d) If $|f| \le M$ for some $M > 0$ then $\left| \int_a^b f \, \mathrm{d}x \right| \le M(b-a)$.

7. (1 pt) Show that the improper integral

$$\int_0^\infty \frac{\sin x}{x} \mathrm{d}x$$

converges, but not absolutely.

8. (1 pt) Give an example of $f \colon \mathbb{R} \to \mathbb{R}$ such that $\int_{-a}^a f \, \mathrm{d}x$ converges as $a \to \infty$, but $\int_{-\infty}^\infty f \, \mathrm{d}x$ does not exist. Give an example f such that $\int_{-a}^{c+a} f \, \mathrm{d}x$ converges as $a \to \infty$ for any fixed $c \in \mathbb{R}$, but $\int_{-\infty}^\infty f \, \mathrm{d}x$ still does not exist.

9. (1 pt) Let $f, g \colon [0,\infty) \to (0,\infty)$ be such that $\int_0^\infty g(x) \, \mathrm{d}x$ diverges. Show that at least one of the integrals

$$\int_0^\infty f(x) g(x) \, \mathrm{d}x, \quad \int_0^\infty \frac{g(x)}{f(x)} \mathrm{d}x$$

diverges. *(Hint: recall the inequality* $2 \le \frac{a}{b} + \frac{b}{a}$*, for* $a, b > 0$*.)*

10. (1 pt) Determine whether (or for which $\alpha \in \mathbb{R}$, or $k \in \mathbb{N}$) each of the following (improper integrals or series) converges.

(a) $\int_0^1 x^k (\log x)^k \, \mathrm{d}x$, where $k \in \mathbb{N}$,
(b) $\int_2^\infty \frac{\mathrm{d}x}{x \log x}$,
(c) $\int_0^1 (-\log x)^\alpha \mathrm{d}x$,
(d) $\sum_{n \ge 3} \frac{1}{n \, \log n \, \log\log n}$.

11. (1 pt) Consider a sequence of functions

$$f_n(x) := \begin{cases} \frac{x}{n} & \text{if } n \text{ is even,} \\ \frac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Show that $f_n \to 0$ pointwise, but not uniformly. Find a uniformly convergent subsequence.

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