MATH 425a PS 12

1. Note that $\int_0^1 f(x)dx = \lim_{n\to\infty}\int_{\frac{1}{2n}}^{1-\frac{1}{2n}} f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n-1}\int_{\frac{2k-1}{2n}}^{\frac{2k+1}{2n}} f(x)dx$.

By MVT, $\hookrightarrow = \lim_{n\to\infty} \sum_{k=1}^{n-1}\int_{\frac{2k-1}{2n}}^{\frac{2k+1}{2n}} f(\xi_{n,k})dx = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} f(\xi_{n,k})$.

Use non-decreasing monotonicity,

$\hookrightarrow \frac{1}{n}\sum_{k=1}^{n-1} f(\frac{2k-1}{2n}) \leq \frac{1}{n}\sum_{k=1}^{n-1} f(\xi_{n,k}) \leq \frac{1}{n}\sum_{k=1}^{n-1} f(\frac{2k+1}{2n})$

We have $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} f(\frac{2k-1}{2n}) = \lim_{m\to\infty}\lim_{mesh\to 0} L(f,P) = \lim_{m\to\infty}\int_{1/m}^{1-1/m} f(x)dx \leq \int_0^1 f(x)dx$

$\leq \lim_{m\to\infty}\int_{1/m}^{1-1/m} f(x)dx = \lim_{m\to\infty}\lim_{mesh\to 0} U(f,P) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} f(\frac{2k+1}{2n})$.

By squeeze theorem the three limits above must agree. Thus,

$\lim_{n\to\infty} \frac{1}{n}(f(\frac{1}{2n}) + f(\frac{3}{2n}) + \cdots + f(\frac{2n-3}{2n})) = \lim_{n\to\infty}\frac{1}{n}(f(\frac{1}{n}) + \cdots + f(\frac{n-1}{n})) = \int_0^1 f(x)dx$.

$\Rightarrow \lim_{n\to\infty} \frac{1}{n}(n)^{\frac{1}{n}} = \frac{1}{e}$.

2. Suppose $\sum_{n\geq 1} f'(n)$ converges. $\because f' \downarrow 0$, $\therefore f' > 0$.

$\because \frac{f'(n)}{f(1)} > \frac{f'(n)}{f(n)}$ as $f(n) > f(1)$ when $n > 1$, $\sum_{n\geq 1} f'(n)/f(1) \geq \sum_{n\geq 1} f'(n)/f(n)$.

$\Rightarrow$ former series converges, $\sum_{n\geq 1} f'(n)/f(n)$ also converges.

Suppose $\sum_{n\geq 1}\frac{f'(n)}{f(n)}$ converges. $\because f'(n)\downarrow, f(n)\uparrow$, we know $\frac{f'(n)}{f(n)}$ non-increasing, non-negative.

$\hookrightarrow \int_0^\infty \frac{f'(n)}{f(n)}dn = [\ln(f(n))]_0^\infty$ exists.

$f'(n)$ also non-increasing, non-negative $\Rightarrow \int_0^\infty f'(n) = [f(n)]_0^\infty \Rightarrow$ this must also exist given above.

Hence $\sum_{n\geq 1} f(n)$ converges.

3. $\frac{a_{n+1}}{s_n \log s_n} = \frac{s_{n+1}-s_n}{s_n\log s_n} = \int_{s_n}^{s_{n+1}} \frac{1}{s_n \log s_n}dx \geq \int_{s_n}^{s_{n+1}}\frac{1}{x\log x}dx$ (because $\frac{1}{s_n\log s_n}$ non-increasing).

We know $\sum_{n\geq 1}\frac{1}{x\ln x}$ diverges $\Rightarrow \sum_{n\geq 1}\frac{a_{n+1}}{s_n\log s_n}$ diverges.

$\frac{a_n}{s_n(\log s_n)^2} = \frac{s_n - s_{n-1}}{s_n(\log s_n)^2} = \int_{s_{n-1}}^{s_n}\frac{1}{s_n(\log s_n)^2}dx \leq \int_{s_{n-1}}^{s_n}\frac{1}{x\log x}dx$ (check this)

we know $\sum_{n\geq 1}\frac{1}{x(\ln x)^2}$ converges $\Rightarrow \sum_{n\geq 1}\frac{a_n}{s_n(\log s_n)^2}$ converges.

4. a) $f_n \rightrightarrows f$: $\exists N_1$ s.t. $|f_n(x) - f(x)| < \frac{\varepsilon}{2}\ \forall n > N_1\ \forall x$.

$g_n \rightrightarrows g$: $\exists N_2$ s.t. $|g_n(x) - g(x)| < \frac{\varepsilon}{2}\ \forall n > N_2\ \forall x$.

$\hookrightarrow |f_n(x)+g_n(x) - (f(x)+g(x))| \leq |f_n(x)-f(x)| + |g_n(x)-g(x)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$, $\forall n > \max\{N_1, N_2\}$, $\forall x$.

$\Rightarrow$ Thus $\{f_n(x) + g_n(x)\} \rightrightarrows f(x)+g(x)$. $\to$ on $(0,1)$.

Counterexample: let $f_n(x) = \frac{1}{n}$, $g_n(x) = \frac{1}{x}$. $\frac{1}{nx}$ not U.C. to 0 (pick $x < \frac{1}{n\varepsilon}$).

b) fix x, $|\frac{1}{nx} - 0| < \varepsilon$ if we pick n so large s.t. $n > \frac{1}{\varepsilon x}$. $\Rightarrow f_n g_n \to fg$ pointwise.

c) ???

7. $\int_{1}^{\infty} \frac{\sin x}{x}\,dx = \lim_{b\to\infty}\int_{1}^{b}\frac{1}{x}\sin x\,dx = \lim_{b\to\infty}[\frac{1}{x}(-\cos x)]_{1}^{b} - \lim_{b\to\infty}\int_{1}^{b}\frac{\cos x}{x^2}\,dx$

Notice $\int_0^\infty \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx$, but $\int_0^1 \frac{\sin x}{x}$ is riemann integrable.

↳ $= -\frac{1}{b}\cos b + \cos 1 - \lim_{b\to\infty}\int_1^b \frac{\cos x}{x^2}dx$, also note that $\frac{\cos x}{x^2} \le \frac{1}{x^2} \Rightarrow \int_1^\infty \frac{\cos x}{x^2}$ convergent.

~~$=0+\cos 1$~~ ↘ 0 as $b\to\infty$ ⇒ Thus $\int_0^\infty \frac{\sin x}{x}dx$ convergent.

$\int_0^\infty |\frac{\sin x}{x}|$ not absolutely convergent: $\int_0^{2\pi N}|\frac{\sin x}{x}|dx = \sum_{n=0}^{N-1}\int_{2\pi n}^{2\pi(n+1)}|\frac{\sin x}{x}|dx \ge \sum_{n=0}^{N-1}\frac{1}{2\pi(n+1)}\int_{2\pi n}^{2\pi(n+1)}|\sin x|dx$

↳ $= \sum_{n=0}^{N-1}\frac{1}{2\pi(n+1)}\cdot 4$

↳ diverges.

9. Apply the fact that $f(x)+\frac{1}{f(x)} \ge 2$ and $g(x)>0$.

↳ $\int_0^\infty f(x)g(x)dx + \int_0^\infty \frac{g(x)}{f(x)}dx \ge \int_0^\infty 2g(x)dx$. ⇒ diverges.

Hence either $\int_0^\infty f(x)g(x)dx$ or $\int_0^\infty \frac{g(x)}{f(x)}dx$ diverges.

10. a) $\int_0^1 (x\log x)^k dx$ = ~~$\lim_{b\to 0}\int_b^1 \log x \cdot x\log x\,dx = [\ldots]_b^1 - \int_b^1$~~

$= \lim_{b\to 0}\int_b^1 x^k(\log x)^k dx = [\frac{x^{k+1}}{k+1}(\log x)^k]_b^1 - \int_b^1 \frac{kx^{k}}{k+1}(\log x)^{k-1}dx$

$= \frac{1}{k+1}\cdot 0 - 0 - [\frac{k}{k+1}[(\frac{1}{k+1})x^{k+1}(\log x)^{k-1} - \frac{x^{k+1}}{(k+1)^2}]]_0^1$ which exists.

b) $\int_2^\infty \frac{1}{x\log x}dx = \lim_{b\to\infty}[\log(\log x)]_2^b \Rightarrow$ DNE. It diverges.

c) $\int_0^1 (-\log x)^a dx$ = converges

d) $\sum_{n\ge 3}\frac{1}{n\log n\log(\log n)}$: non-increasing, non-negative, check $[\ln(\ln(\ln x))]_3^\infty \Rightarrow$ diverges.

11. ~~$|\frac{x}{n}-\frac{1}{n}| = \frac{x-1}{n}$ ∀ε>0, pick~~ given x.

$d(f_n(x), 0) = \frac{x}{n}$ or $\frac{1}{n}$. Given $\varepsilon>0$, pick sufficiently large n s.t. $\frac{x}{n}$ or $\frac{1}{n} < \varepsilon \Rightarrow f_n \to 0$ pointwise.

Why not uniformly convergent: Suppose U.C. Then fix ε, ~~we have~~ $\exists N\in\mathbb{N}$ s.t. $|\frac{x}{n}-0|<\varepsilon$ $\forall n>N$, $x\in X$.

Pick any $x > n\varepsilon$ ~~cont~~ results in a contradiction.

Example of ≠ U.C. subsequence: $\{f_n(x)\}_{n \text{ is odd}}$.