Solutions to Problem Set 12

Math 425a, Fall 2021

November 30, 2021

1. First fix some $n > 1$ and choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{n}$. Now look at the interval $[\varepsilon, 1 - \frac{1}{n}]$. From the given conditions, $f \in \mathcal{R}$ on this interval. Consider the partition $\{\varepsilon, \frac{1}{n}, \frac{2}{n}, \cdots, 1 - \frac{1}{n}\}$. We recognize that

$$f\left(\frac{1}{n}\right)\left(\frac{1}{n} - \varepsilon\right) + \frac{f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n}$$

is the upper Riemann sum for f on this partition, since f in nondecreasing. Therefore (taking inf over all partitions)

$$\int_{\varepsilon}^{1-\frac{1}{n}} f(x)dx \leq f\left(\frac{1}{n}\right)\left(\frac{1}{n} - \varepsilon\right) + \frac{f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n}.$$

Letting $\varepsilon \to 0$ and noting that $\int_0^1 f(x)dx$ exists (hence, so does $\int_0^{1-\frac{1}{n}} f(x)dx$) we get

$$\int_0^{1-\frac{1}{n}} f(x)dx \leq \frac{f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n} \tag{1}$$

Similarly looking at the interval $[\frac{1}{n}, 1-\varepsilon]$ and recognising $\frac{f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n-2}{n})}{n}+f\left(\frac{n-1}{n}\right)\left(\frac{1}{n} - \varepsilon\right)$ is the lower Riemann sum, we get the inequality $\int_{\frac{1}{n}}^{1-\varepsilon} f(x)dx \geq \frac{f(\frac{1}{n})+f(\frac{2}{n})+\cdots+f(\frac{n-2}{n})}{n} + f\left(\frac{n-1}{n}\right)\left(\frac{1}{n} - \varepsilon\right)$. Again letting $\varepsilon \to 0$ we get

$$\int_{\frac{1}{n}}^{1} f(x)dx \geq \frac{f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n} \tag{2}$$

Combining both (1) and (2) we get

$$\int_0^{1-\frac{1}{n}} f(x)dx \leq \frac{f(\frac{1}{n}) + f(\frac{2}{n}) + \cdots + f(\frac{n-1}{n})}{n} \leq \int_{\frac{1}{n}}^{1} f(x)dx$$

Now taking limits as $n \to \infty$ and applying Squeeze Theorem (Cor. 5.9) gives us the desired result.

For the second part, note that $f(x) = \ln x$ satisfies the conditions of the problem with $\int_0^1 \ln x dx = -1$. Therefore using our our result,

$$\lim_{n\to\infty} \frac{\ln(\frac{1}{n}) + \cdots + \ln(\frac{n-1}{n})}{n} = -1.$$

Taking e to this power, and recalling that the exponential function is continuous, we obtain

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e^{-1},$$

as requried.

2. We use the Integral test for convergence of series on this problem. First assume $\sum_{n\geq 1} f'(n)$ converges. Note that since f' is decreasing function and positive from hypothesis we can use the Integral test to conclude that our assumption implies $\int_1^\infty f'(x)dx$ converges. But $\int_1^\infty f'(x)dx = \lim_{b\to\infty} \int_1^b f'(x)dx = \lim_{b\to\infty} f(b) - f(1)$. Therefore the convergence of $\sum_{n\geq 1} f'(n)$ implies that $\lim_{b\to\infty} f(b)$ is finite. Let this limit equal L. Now note that

$$\int_1^\infty \frac{f'(x)}{f(x)} = \lim_{b\to\infty} \int_1^b \frac{f'(x)}{f(x)} = \lim_{b\to\infty} \ln(f(b)) - \ln(f(1)) = \ln L - \ln f(1) < \infty \qquad (3)$$

.

Next observe that since f' is always positive, f is a nondecreasing function and it is positive from given conditions, which implies $\frac{f'(x)}{f(x)}$ is a positive, decreasing function. So we can apply the integral test to this function too and using (3) we conclude $\sum_{n\geq 1} \frac{f'(n)}{f(n)}$ converges.

Now if we assume $\sum_{n\geq 1} \frac{f'(n)}{f(n)}$ converges then using (3) again we conclude that $\lim_{b\to\infty} f(b)$ is finite which will imply $\int_1^\infty f'(x)dx$ converges and hence $\sum_{n\geq 1} f'(n)$ too.

Therefore, one of the two series converges if and only if the other one does too which is just another way of saying that they both converge or both diverge.

3. Note that $a_{n+1} = S_{n+1} - S_n = \int_{S_n}^{S_{n+1}} 1\, dx$ (by Lem. 11.4(f)), and so

$$\sum_{n\geq 1} \frac{a_{n+1}}{S_n \log S_n} = \underbrace{\sum_{n=1}^{N} \frac{a_{n+1}}{S_n \log S_n}}_{=:C_N} + \sum_{n>N} \int_{S_n}^{S_{n+1}} \frac{1}{S_n \log S_n} dx \geq C_N + \sum_{n>N} \int_{S_n}^{S_{n+1}} \frac{1}{x \log x} dx$$

$$= C_N + \int_{S_{N+1}}^{\infty} \frac{1}{x \log x} dx = \infty,$$

where N is such that $S_N \geq 2$ (so that $x \log x$ is increasing for $x > S_N$). Similarly

$$\sum_{n\geq 1} \frac{a_n}{S_n(\log S_n)^2} = C'_N + \sum_{n>N} \int_{S_{n-1}}^{S_n} \frac{1}{S_n(\log S_n)^2} dx \leq C'_N + \sum_{n>N} \int_{S_{n-1}}^{S_n} \frac{1}{x(\log x)^2} dx$$
$$= C'_N + \int_{S_N}^{\infty} \frac{1}{x(\log x)^2} dx < \infty,$$

where $C'_N := \sum_{n=1}^{N} \frac{a_n}{S_n(\log S_n)^2}$.

4. (a) Note that

$$\begin{aligned}\sup_{x\in X} |(f_n(x) + g_n(x)) - (f(x) + g(x))| &\leq \sup_{x\in X}(|f_n(x) - f(x)| + |g_n(x) - g(x)|) \\ &\leq \sup_{x\in X} |f_n(x) - f(x)| + \sup_{x\in X} |g_n(x) - g(x)| \\ &\longrightarrow 0 \ \text{ as n} \to \infty\end{aligned}$$

where we used PS1.8 in the second line and the last statement holds because $f_n \rightrightarrows f$ and $g_n \rightrightarrows g$.

To provide a counterexample, take $f_n(x) = \frac{1}{x}$ (the constant sequence) and let $g_n(x) = \frac{1}{n}$ (the sequence of constant functions) where $x \in (0, 1)$. It is clear that the pointwise limit of f_n is the function $f(x) = \frac{1}{x}$ (this is also the uniform limit because f_n is the constant sequence) and that of g_n is the function $g(x) = 0$. Next note that that the convergence of g_n is uniform too (as there is no x appearing at all). Now note that $f_n(x)g_n(x) = \frac{1}{nx}$ also converges to the zero function point wise. But,

$$\sup_{x\in(0,1)} |f_n(x)g_n(x)| = \sup_{x\in(0,1)} \frac{1}{nx} > \frac{1}{n \cdot \frac{1}{n^2}} = n$$

which proves convergence is not uniform.

(b) $f_n \to f$ point wise means that the sequence $f_n(x) \to f(x)$ for every $x \in X$. Similarly, $g_n(x) \to g(x)$ for every $x \in X$. The desired result follows from Cor. 7.4.

(c) Let M be such that both $|f|, |g| < M$. Note that this implies that both f_n and g_n are bounded by $M + 1$ for sufficiently large n (i.e. the n from the definition of uniform convergence for $\varepsilon := 1$). Thus

$$\begin{aligned}\sup_{x\in X} |f_n(x)g_n(x) - f(x)g(x)| &= \sup_{x\in X} |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\ &\leq \sup_{x\in X} \big(|f_n(x)||g_n(x) - f(x)g(x)| + |g(x)||f_n(x) - f(x)|\big) \\ &\leq (M + 1) \sup_{x\in X} |g_n(x) - g(x)| + M \sup_{x\in X} |f_n(x) - f(x)| \\ &\longrightarrow 0 \ \text{ as n} \to \infty,\end{aligned}$$

proving the claim.

5. We use Taylor expansion with the remainder in integral form. Expanding around 0 we get,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)x^k}{k!} + \int_0^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt$$

Putting in the values for the derivatives, we get therefore,

$$f(x) = \int_0^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt$$

which gives us, for $0 < x < 1$,

$$|f(x)| \leq \int_0^x |f^{(n)}(t)|\frac{|x-t|^{n-1}}{(n-1)!}dt \leq \frac{1}{(n-1)!}\int_0^1 |f^{(n)}(t)|dt$$

Integration in x gives

$$\int_0^1 |f(x)|dx \leq \frac{1}{n!}\int_0^1 \left(\int_0^1 |f^{(n)}(t)|dt\right)dx = \frac{1}{n!}\int_0^1 |f^{(n)}(t)|dt$$

which is the desired inequality.

6. Let $\alpha, \beta \in (a, b)$ be such that $\alpha < \beta$. Then since both $f, g \in \mathcal{R}$ on $[\alpha, \beta]$ we have by Lem. 11.4 that,

   (a) $\int_\alpha^\beta (f + cg)dx = \int_\alpha^\beta f dx + c\int_\alpha^\beta gdx$

   (b) If $f \leq g$ then $\int_\alpha^\beta f dx \leq \int_\alpha^\beta gdx$

   (c) $\int_\alpha^\beta f dx = \int_\alpha^c f dx + \int_c^\beta f dx$ for all $c \in (\alpha, \beta)$

   (d) $|\int_\alpha^\beta f dx| \leq \int_\alpha^\beta |f|dx \leq M(\beta - \alpha)$

   Now taking limits as $\beta \to b^-$ and $\alpha \to a^+$ in the equations above gives us the required results. Note that we are allowed to split the limits in the LHS of part (a) as we know that both improper integrals $\int_a^b f dx$ and$\int_a^b gdx$ exist.

7. Integration by parts gives us,

$$\int_1^\infty \frac{\sin x}{x}dx = \left[\frac{-1}{x}\cos x\right]_1^\infty - \int_1^\infty \frac{1}{x^2}\cos xdx = \cos(1) - \int_1^\infty \frac{1}{x^2}\cos xdx$$

   Now note that $\int_1^\infty \frac{1}{x^2}|\cos x|dx \leq \int_1^\infty \frac{1}{x^2}dx < \infty$, which tells us that $\int_1^\infty \frac{1}{x^2}\cos xdx$ converges by Lem. 12.5. Therefore $\int_1^\infty \frac{\sin x}{x}dx$ converges too. Combining this with the fact that $\frac{\sin x}{x}$ is continuous on $[0, 1]$ (after redefining by 1 at 0, which tells us $\int_0^1 \frac{\sin x}{x}dx < \infty$ due to PS11.2) gives us that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent.

Now we show that the integral does not converge absolutely. Note for all $N > 1$,

$$\begin{aligned}
\int_0^{2\pi N} \left|\frac{\sin x}{x}\right| dx &= \sum_{k=0}^{N-1} \int_{2\pi k}^{2\pi(k+1)} \left|\frac{\sin x}{x}\right| dx \\
&\geq \sum_{k=0}^{N-1} \frac{1}{2\pi(k+1)} \int_{2\pi k}^{2\pi(k+1)} |\sin x| dx \\
&= \sum_{k=0}^{N-1} \frac{1}{2\pi(k+1)} \int_0^{2\pi} |\sin x| dx \\
&= \sum_{k=0}^{N-1} \frac{1}{2\pi(k+1)} \\
&\geq \frac{1}{4\pi} \sum_{k=1}^{N-1} \frac{1}{k}
\end{aligned}$$

Since the RHS diverges as $N \to \infty$ (by Ex. 6.8), we conclude that $\int_0^\infty |\frac{\sin x}{x}| dx$ diverges too.

8. The point is to notice that the integrals in question have upper and lower limits that tend to infinity with the same rate (i.e. a linear rate), so one can take a function that is not integrable as $x \to \infty$, but will be cancelling out with the corresponding part on the negative side when the two integration limits go to $\pm\infty$ with the same rate. Let, for example

$$f(x) := \begin{cases} (x+1)^{-1} & x \geq 0, \\ -(1-x)^{-1} & x < 0. \end{cases}$$

Let $c \in \mathbb{R}$ (WLOG $c \geq 0$; the case $c < 0$ is analogous). Then

$$\begin{aligned}
\int_{-a}^{c+a} f dx &= \underbrace{-\int_{-a}^{0} \frac{dx}{1-x} + \int_0^a \frac{dx}{1+x}}_{=0} + \int_a^{c+a} \frac{dx}{1+x} = \log(1+c+a) - \log(1+a) \\
&= \log \frac{1+c+a}{1+a} \to 0
\end{aligned}$$

as $a \to \infty$ (since log is a continuous function, and its argument tends to 1).

However,

$$\int_0^a f dx = \int_0^a \frac{dx}{1+x} = \log(1+a) \to \infty,$$

as $a \to \infty$, and so $\int_{-\infty}^{\infty} f dx$ does not converge.

This question illustrates the comment from Def. 12.4 that both limits at ∞ and $-\infty$ must exist separately to be define the improper integral "$\int_{-\infty}^{\infty}$".

9. Note that for any $B > 0$, we have by the given inequality,

$$\int_0^B f(x)g(x)dx + \int_0^B \frac{g(x)}{f(x)}dx = \int_0^B \left(f(x)g(x) + \frac{g(x)}{f(x)}\right)dx \geq 2\int_0^B g(x)dx.$$

Note that we integrate nonnegative functions in each of the integrals above. As $B \to \infty$ the RHS diverges, which implies that at least one of the integrals on the LHS must diverge too.

10. (a) Recall that $\lim_{x\to 0^+} x\log x = 0$ (by de l'Hôpital rule, Thm. 9.20), which tells us that $(x\log x)^k \to 0$ as $x \to 0^+$ for all $k \in \mathbb{N}$. Therefore $(x\log x)^k$ has a continuous extension to $[0,1]$ which tells us (by Theorem 10.17 and PS11.2) that the given integral is finite for all $k \in \mathbb{N}$.

(b) Note that $\int_2^B \frac{dx}{x\log x} = \log\log B - \log\log 2$. However, $\lim_{B\to\infty} \log\log B = +\infty$ which tells us that the integral diverges.

(c) The substitution $-\log x = t$ gives us

$$\int_0^1 (-\log x)^\alpha dx = \int_0^\infty t^\alpha e^{-t}dt = \int_0^1 t^\alpha e^{-t}dt + \int_1^\infty t^\alpha e^{-t}dt \tag{4}$$

Note now that for all $\alpha \in \mathbb{R}$, $\lim_{t\to\infty} \frac{t^\alpha e^{-t}}{t^{-2}} = 0$ which implies there exists $M > 0$, such that for all $t > M$ we have $\frac{t^\alpha e^{-t}}{t^{-2}} \leq \frac{1}{2}$, which in turn gives us, $\int_M^\infty t^\alpha e^{-t}dt \leq \int_M^\infty t^{-2}dt < \infty$. So we have that the second integral in (4) always converges for any α.

For the first integral, note that $\lim_{t\to 0^+} \frac{t^\alpha e^{-t}}{t^\alpha} = 1$, and so (since e^{-t} is decreasing), we have $\frac{1}{e} \leq \frac{t^\alpha e^{-t}}{t^\alpha} \leq 1$ on $(0,1]$. This in turn implies $\int_0^\delta t^\alpha e^{-t}dt$ converges if and only if $\int_0^\delta t^\alpha dt$ converges which holds if and only if $\alpha > -1$.

Combining everything gives us that the whole integral converges if and only if $\alpha > -1$.

(d) Clearly $f(x) = x\log x\log\log x$ is an increasing and positive function for $x \geq 3$ (recall that $\log\log x > 0$ for $x > e$, so in particular for $x > 3$). So by the integral test (Thm 12.7) we have that $\sum_{n\geq 3} \frac{1}{n\log n\log\log n}$ converges if and only if $\int_3^\infty \frac{1}{x\log x\log\log x}dx$ converges. But

$$\int_3^\infty \frac{1}{x\log x\log\log x}dx = \lim_{B\to\infty} \int_3^B \frac{1}{x\log x\log\log x}dx =$$
$$\lim_{B\to\infty} \log\log\log B - \log\log\log 3 = +\infty$$

giving us that the series does not converge.

11. Given x we have $|f_n(x)| \leq \max(|x|, 1)/n \to 0$ as $n \to \infty$, and so $f_n \to 0$ pointwise.

The convergence is not uniform because $\sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{|x|}{n} = \infty$ for even n, which evidently does not converge to 0.

The subsequence of f_n's consisting of only odd n values converges uniformly (to 0) because

$$\sup_{x\in\mathbb{R}} |f_{2n+1}(x)| = 1/(2n+1) \to 0$$

as $2n+1 \to \infty$. Note that for the sequence of constant functions there is no difference between the two notions of convergence.