**Math 425A Problem Sheet 13 (due 9am on Monday, 29th Nov)**

***Essential problems***

**1.** (3 pts) Let $(X, d)$ be metric space. Suppose that $f_n\colon X \to \mathbb{R}$ is such that each $f_n$ is uniformly continuous, and that $f_n \rightrightarrows f$ as $n \to \infty$ for some $f\colon X \to \mathbb{R}$.

(a) Show that $f$ is uniformly continuous. Why does it not follow from Cor. 13.2 or from Cor. 13.3?

(b) Suppose further that each of $f_n$ is Lipschitz continuous. Does this imply that $f$ is Lipschitz continuous?

(c) Suppose further that all $f_n$'s are Lipschitz continuous with the same constant, i.e. there exists $C > 0$ such that $|f_n(x) - f_n(y)| \leq Cd(x, y)$ for all $x, y \in X$, $n \geq 1$. Show that $f$ is Lipschitz continuous with the same constant $C$.

*(Comments: the last two parts of this problem illustrate the principle we keep seeing in the last couple of weeks: adding a degree of uniformity to a sequence of functions gives better properties of the limit function (cf. e.g. Cor. 13.2, Thm. 13.7, Thm. 13.12, etc.); moreover (spoiler alert) one can think of parts (b) and (c) as an "appetizer" to the Arzelà-Ascoli theorem, which we will see soon (and which you should not use to solve this problem).)*

**2.** (2 pts) Show that all assumptions in Thm. 13.4 are essential. Namely, consider all the assumptions (1) $f_n \geq f_{n+1}$, (2) $f_n$'s are continuous, (3) $f$ is continuous, (4) the domain $K$ is compact. For each of the assumptions (1)-(3) find a sequence $(f_n)$, and $f$, that do not satisfy the given assumption, but do satisfy the remaining three assumptions, and such that $f_n \to f$ pointwise, but not uniformly.
*(Comment: As for assumption (4), we have already found such counterexample in Ex. 13.5; also hint: consider some polynomials, or piecewise affine functions.)*

**3.** (3 pts) Suppose that, given $n \geq 1$, $f_n \in C([0, 1])$ is nondecreasing. Suppose that, for some $f \in C([0, 1])$, $f_n(q) \to f(q)$ as $n \to \infty$ for every $q \in \mathbb{Q} \cap [0, 1]$. Show that $f_n \rightrightarrows f$ on $[0, 1]$.
*(Comment: This problem is an example of another (rare) situation where pointwise convergence implies uniform convergence, cf. Thm. 13.4. In fact in this setting one does not even need convergence pointwise, but merely convergence at rational points.)*

**4.** (2 pts) Suppose that $f_n \in C^1([a, b])$ are such that $f_n' \rightrightarrows g$ as $n \to \infty$ for some $g\colon [a, b] \to \mathbb{R}$ and that $f_n(x_0) \to \alpha$ for some $\alpha \in \mathbb{R}$, $x_0 \in [a, b]$.

(a) Show that $f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,\mathrm{d}t$ for every $x \in [a, b]$, $n \geq 1$.

(b) Deduce that there exists $f \in C([a, b])$ such that $f_n \rightrightarrows f$ on $[a, b]$ as $n \to \infty$.

(c) Deduce that $f \in C^1([a, b])$ and $f' = g$.

*(Comment: Note in this problem we give a different/faster proof of Thm. 13.7 in the case when $f_n'$ is continuous for each $n$.)*

***Additional problems***

**5.** (1 pt) Does $f_n(x) := \arctan \frac{2x}{x^2+n^3}$ converge uniformly on $\mathbb{R}$ (to some function)?

**6.** (1 pt) Give an example of a sequence of functions $(f_n)$ defined on $\mathbb{R}$, such that each $f_n$ is discontinuous at every point (i.e. $f$ is not continuous at $x$, for every $x \in \mathbb{R}$), but $f_n \rightrightarrows f$ as $n \to \infty$ for some $C^\infty$ function $f\colon \mathbb{R} \to \mathbb{R}$. Why does it not contradict Thm. 13.1?

**7.** (1 pt) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and $f'$ is uniformly continuous (on $\mathbb{R}$). Show that

$$\frac{f(x+1/n) - f(x)}{1/n} \rightrightarrows f'(x)$$

as $n \to \infty$. Show that this claim might be false if $f'$ is not uniformly continuous.

**8.** (1 pt) Show that the series

$$\sum_{n\geq 1} (-1)^n \frac{n+x^2}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any $x$.

**9.** (1 pt) Let $(X, d)$ be a metric space. Let $f_n, g_n\colon X \to \mathbb{R}$ be such that

$$S_m := \sum_{n=1}^{m} f_n$$

is bounded uniformly in $m$ (i.e. there exists $M > 0$ such that $|S_m(x)| \leq M$ for all $x \in X$, $m \geq 1$). Suppose that $g_n \geq g_{n+1}$ for all $n \geq 1$ and that $g_n \rightrightarrows 0$ as $n \to \infty$. Show that $\sum_{n\geq 1} f_n g_n$ converges uniformly on $X$. *(Hint: Recall Thm. 6.18.)*

**10.** (1 pt) Consider $f_n(x) := \frac{x}{1+n^2x^2}$ for $x \in [-1, 1]$. Determine whether the sequence $(f_n')_{n\geq 1}$ converges uniformly. How about $(f_n)_{n\geq 1}$?

**11.** (1 pt) Find

$$\lim_{n\to\infty} \int_0^1 nx(1-x^2)^n \mathrm{d}x.$$

Can you use Thm. 13.12 to find this limit?

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