MATH 425a PS 13

1. a) $f_n \rightrightarrows f \Rightarrow \forall \varepsilon>0, \exists N$ s.t. $\sup_{x\in X}|f_n(x)-f(x)|<\varepsilon$ if $n\geq N$.

f_n uniformly continuous: $\forall \varepsilon>0, \exists \delta$ s.t. $|f_n(x)-f_n(y)|<\varepsilon$ if $|x-y|<\delta$.

let $x, y \in X$. Pick sufficiently large N s.t. $|f(x)-f_n(x)|, |f_n(y)-f(y)|<\frac{\varepsilon}{3}$.

↳ This is achievable as $\sup_{x\in X}|f_n(x)-f(x)|<\frac{\varepsilon}{3}$ .... Now pick δ' s.t. ~~such~~ $|f_n(x)-f_n(y)|<\frac{\varepsilon}{3}$ if $|x-y|<\delta'$.

$\Rightarrow \forall \varepsilon>0, |f(x)-f(y)| \leq \cancel{|f_n(x)} |f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)|<\frac{\varepsilon}{3}\cdot 3=\varepsilon$. (if sufficiently small δ').
↳ $\forall x, y \in X$.

Hence f is uniformly continuous.

Why 13.2 fails: uniform convergence preserves continuity, NOT uniform continuity.

b) f_n Lipschitz continuous $\Rightarrow \exists L_n>0$ s.t. $|f_n(x)-f_n(y)|\leq L_n|x-y|$ $\forall x, y\in X$.

~~We know $|f(x)-f(y)| \leq L_n|x-y|+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=L_n|x-y|+\varepsilon$.~~

~~since ε is an arbitrary variable, pick L_n' s.t. $L_n'|x-y|<\varepsilon$ and f is thus Lipschitz as well.~~ >0

$|f(x)-f(y)|\leq L_n|x-y|+2\varepsilon$.

~~⇐~~ To prove Lipschitz continuity, $|f(x)-f_n(x)|$ must $<$ ~~(as well as ε~~ $L_n'|x-y|$ $\forall n\geq N$, $L_n'>0$.

Hence N depends on the choice of x and y. We can't find N that makes the above inequality hold $\forall x, y \in X$. f is thus not Lipschitz continuous. (Think about $\sup\{L_n\}$ blows up).

↳ can only guarantee $|f(x)-f(y)|<\infty$.

c) $|f(x)-f(y)|=\lim_{n\to\infty}|f_n(x)-f_n(y)|\leq C|x-y|$ if the Lipschitz constant has an upper bound.

Hence f is Lipschitz with constant C (actually, only the largest L_n needs to be C ...)

2. (1) let f not monotone non-increasing: $f_n(x)=\begin{cases}0 & 0\leq x<\frac{1}{2n}\\ 4nx-2 & \frac{1}{2n}\leq x<\frac{3}{4n}\\ 4-4nx & \frac{3}{4n}\leq x<\frac{1}{n}\\ 1 & \frac{1}{n}\leq x\leq 1\end{cases}$ on $[0,1]$.

(2) let f_n not continuous: $f_n(x)=\begin{cases}1 & 1-\frac{1}{n}\leq x<1\\ 0 & \text{otherwise on } [0,1]\end{cases}$

(3) let f not continuous: consider $f_n(x)=x^n$ on $[0,1]$.

4. ~~5~~ a) Since $f_n\in C'([a,b])$, f_n is Riemann integrable.

Hence $f_n(x)-f_n(x_0)=\int_{x_0}^{x}f_n'(t)dt$ $\forall x\in[a,b]$ as a result of FTC.

b) $f(x)=a+\int_{x_0}^{x}g(t)dt$ is the uniform limit of f_n.

$\forall \varepsilon, \exists N_1$ s.t. $|f_n(x_0)-f(x_0)|<\frac{\varepsilon}{2}$ $\forall n\geq N_1$, also

$\because f_n'\to g$ $\exists N_2$ s.t. $\sup_{x\in[a,b]}|f_n'(x)-g(x)|<\frac{\varepsilon}{2(b-a)}$. Take $N:=\max\{N_1, N_2\}$.

If $n\geq N$, $\Rightarrow \forall x\in[a,b]$, $|f_n(x)-f(x)|=|a+\int_{x_0}^{x}g(t)dt-f_n(x_0)-\int_{x_0}^{x}f_n'(t)dt|$

$\leq |a-f_n(x_0)|+\int_{x_0}^{x}|g-f_n'(t)|dt|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.

Thus, $f_n\rightrightarrows f$ and $f\in C([a,b])$.

c) g is continuous because $f_n'\rightrightarrows g$. By FTC it follows that $f'=g$, hence $f\in C'([a,b])$.

5. Yes, $f(x)=0$.

6. let $f_n(x)=\begin{cases} \frac{1}{n} & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q}. \end{cases}$

10. $f_n'(x)=\frac{1+n^2x^2-n^2x}{(1+n^2x^2)^2} \Rightarrow$ No, does not converge uniformly.
$f_n(x)$: Yes, converges uniformly.

11. $\lim_{n\to\infty}\int_0^1 nx(1-x^2)^n dx = \lim_{n\to\infty} n\int_0^1 (-\frac{1}{2})(-2x)(1-x^2)^n dx = \lim_{n\to\infty} n\left[(-\tfrac{1}{2})\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1 = \lim_{n\to\infty}\frac{n}{2(n+1)}=\frac{1}{2}$.
Theorem 13-12: no, because $nx(1-x^2)^n$ does not converge uniformly on $[0,1]$.

3. f continuous on compact domain $[0,1] \Rightarrow$ It's uniformly continuous.
$\Rightarrow \exists\delta>0$ s.t. $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\frac{\varepsilon}{2}$ $\forall x,y$.
In particular, ~~we def~~ $\exists$ a partition $\{x_0=0, x_1, \dots x_{n-1}, x_n=1\}$ s.t. $|x_i-x_{i-1}|=\frac{1}{n}$.
This means these endpoints must be rational. Also, $\forall x,y\in[x_{i-1},x_i]$, $|f(x)-f(y)|<\frac{\varepsilon}{2}$.
Now, since $\{x_0,\dots x_n\}$ finite, $\exists N$ s.t. $|f_n(x_i)-f(x_i)|<\varepsilon$ $\forall n\geq N$ and $0\leq i\leq n$.
↳ we invoke the fact that $f_n(q)\to f(q)$ when $q\in\mathbb{Q}$.
This N satisfies uni. convergence. Let $t\in[0,1]$. WLOG let $t\in[x_{i-1},x_i]$.
Then, if $n\geq N$, $|f_n(t)-f(t)|\leq|\max\{f(x_i),f_n(x_i)\}-\min\{f(x_{i-1}),f_n(x_{i-1})\}|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$
and thus $f_n \rightrightarrows f$ on $[0,1]$.