Homework Assignment Solution 13

November 30, 2021

1.(a) Let $\epsilon > 0$ be given. First we choose n so that $\sup_x |f(x) - f_n(x)| < \epsilon/3$ (we can do so since $f_n \rightrightarrows f$), and then we choose $\delta > 0$ such that $|f_n(x) - f_n(y)| < \epsilon/3$ for every x, y satisfying $d(x, y) < \delta$ (we can do this since f_n is uniformly continuous). Then for every $x, y \in X$ with $d(x, y) < \delta$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f(y) - f_n(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that f is uniformly continuous on X. Corollaries 13.2 and 13.3 are not applicable, since Cor. 13.2 only says that the limit function is continuous, and Cor. 13.3 is not applicable since X is not necessarily compact. (If X were compact, since f is continuous, it would then be uniformly continuous.)

1.(b) No. Let $X := [0, 1]$ with the usual Euclidean metric, and let $f_n := \sqrt{x + 1/n}$. Then each f_n is differentiable with the derivative

$$f_n'(x) = \frac{1}{2\sqrt{x + 1/n}} \leq \frac{\sqrt{n}}{2}.$$

Therefore (as in (9.2) in the lecture notes) by the MVT, for every $x, y \in [0, 1]$ with $x < y$, there exists $z \in (x, y)$ such that

$$|f_n(x) - f_n(y)| = |f_n'(z)||x - y| \leq \frac{\sqrt{n}}{2}|x - y|,$$

showing that each f_n is Lipschitz.

Let $\epsilon > 0$ be given. Since $x \in [0, 1]$, if we choose $N > 1/\epsilon^2$, then for every $n > N$,

$$\left|\sqrt{x} - \sqrt{x + 1/n}\right| = \left|\frac{1/n}{\sqrt{x} + \sqrt{x + 1/n}}\right| \leq \left|\frac{1/n}{\sqrt{1/n}}\right| = \frac{1}{\sqrt{n}} < \epsilon.$$

Hence $f_n \to \sqrt{x}$ uniformly. Yet $\sqrt{x}$ is not a Lipschitz function on $[0, 1]$ by Ex. 8.14.

1.(c) Taking the limit $n \to \infty$ in $|f_n(x) - f_n(y)| \le Cd(x, y)$ (which is allowed since uniform convergence gives pointwise convergence) shows that f is Lipschitz continuous with the same Lipschitz constant.

2. First consider assumption (1). Take

$$f_n(x) := \begin{cases} nx & \text{if } 0 \le x \le 1/n, \\ 1 - nx & \text{if } 1/n \le x \le 2/n, \\ 0 & \text{if } 2/n \le x \le 1. \end{cases}$$

on $[0, 1]$ with the usual Euclidean metric. Then $f_n(x)$ is continuous on $[0, 1]$ (which is a compact set), and it converges pointwise to the continuous function $f(x) = 0$ for the following reason: Since $f_n(0) = 0$ for every n, $f(0) = 0$. Also, for every $t > 0$, there exists N such that for every $n > N$, $2/n < t$, hence $f_n(t) = 0$ for all $n > N$.

However, convergence is not uniform since $\sup_x |f(x) - f_n(x)| = \sup_x |f_n(x)| = 1$ for every n, because $f_n(1/n) = 1$ (Thm.12.12).

Now consider assumption (2). We give two examples. One is $f_n(x) := 1$ for $x \in (0, 1/n)0$ and 0 for $x \in \{0\} \cup [1/n, 1]$. Similarly as above $f_n \to f$ pointwise, and the domain $[0, 1]$ is compact and $\sup_x |f_n(x) - f(x)| = 1$ for every n, where $f(x) := 0$. Also $f_n \ge f_{n+1}$.

For the second example let $X := \{x_1, x_2, \dots\}$ be an enumeration of rationals in $[0, 1]$, and define $Q_n := \{x_i \in X : i \ge n\}$. Consider the sequence functions

$$f_n(x) := \begin{cases} 1 & \text{if } x \in Q_n, \\ 0 & \text{otherwise}, \end{cases}$$

defined on $[0, 1]$ with the usual Euclidean metric. Then f_n converges pointwise to the constant function $f(x) = 0$, because for every fixed t, if t is irrational, $f_n(t) = 0$ for every n, and if it is rational, it equals x_N for some $N \in \mathbb{N}$, so $f_n(t) = 0$ for all $n > N$. Furthermore, the sequence is a monotone sequence since $Q_n \subset Q_{n+1}$, meaning $f_{n+1}(x) = 0$ at every point where $f_n(x) = 0$. However, convergence is not uniform since $\sup_x |f - f_n| = \sup_x |1 - f_n| = 1$ for every n, because f_n will attain the value of 0 for some x_j with $j > n$.

Now consider assumption (3). Let

$$f_n(x) := x^n$$

defined on $[0, 1]$ with the usual Euclidean metric. Then $f_n(x) \to f(x)$ as $n \to \infty$, where $f(x) := 0$ for $x < 1$ and $f(1) := 1$. Since $x^n \ge x^{n+1}$ for every $x \in [0, 1]$, the sequence is nonincreasing.

However, convergence is not uniform because, for every n, $\sup_x |f_n(x) - f(x)| = 1$ by letting $y_k := 1 - 1/k$ (then $f_n(y_k) \to 1$ as $k \to \infty$).

3. Let $\varepsilon > 0$. Since f is uniformly continuous (by Thm. 8.11) there exists $\delta > 0$ such that

$$|f(x) - f(y)| \le \varepsilon/2 \text{ if } |x - y| \le \delta. \tag{1}$$

Let $N \in \mathbb{N}$ be such that $N > 1/\delta$ and let $x_k := k/N$ for $k = 0, ..., N$. (Note that $x_k \in \mathbb{Q}$ for every k.) By assumption $f_n \to f$ pointwise on rational numbers, so in particular $f_n(x_k) \to f(x_k)$ for every $k = 0, ..., N$. Since there are only finitely many k's, we can choose n_0 such that

$$|f_n(x_k) - f(x_k)| \leq \varepsilon/2 \text{ for every } n \geq n_0 \text{ and } k = 0, ..., N. \tag{2}$$

We now show that $|f_n(x) - f(x)| \leq \varepsilon$ for such n's and all $x \in [0,1]$(which proves uniform convergence by taking $\sup_x$).

Indeed given $x \in [0,1]$ let x_k be such that $x \leq x_k \leq x + \delta$ (which is possible due to the choice of N). Then

$$f(x) \geq f(x_k) - \varepsilon/2 \geq f_n(x_k) - \varepsilon \geq f_n(x) - \varepsilon,$$

where we used (1) in the first inequality, (2) in the second, and monotonicity of f_n in the third inequality.

Analagously we can choose $x_k \in [x - \delta, x]$ and obtain

$$f(x) \leq f_n(x) + \varepsilon.$$

Altogether $|f(x) - f_n(x)| \leq \varepsilon$, as required.

4.(a) This follows from the Fundamental Theorem of Calculus, Thm. 11.11.

4.(b) Let

$$f(x) := \alpha + \int_{x_0}^{x} g(t)dt$$

Then $f_n \rightrightarrows f$ on $[a,b]$ as $n \to \infty$, as

$$\begin{aligned} |f(x) - f_n(x)| &= \left| \alpha + \int_{x_0}^{x} g(t)dt - f_n(x_0) - \int_{x_0}^{x} f_n'(t)dt \right| \\ &\leq |\alpha - f_n(x_0)| + \left| \int_{x_0}^{x} (g(t) - f_n'(t))dt \right| \\ &\leq |\alpha - f_n(x_0)| + (x - x_0) \sup_x |g(x) - f_n'(x)| \\ &\leq |\alpha - f_n(x_0)| + (b - a) \sup_x |g(x) - f_n'(x)|, \end{aligned}$$

where we used Thm. 11.5 and Lem. 11.4(b) and (f) in the third line.

Given ϵ choose N large enough so that $|\alpha - f_n(x_0)| < \epsilon/2$ and $\sup_x |g(x) - f_n'(x)| < \epsilon/2(b-a)$ (we can do this because $f_n(x_0) \to \alpha$ and $f_n' \rightrightarrows g$). Thus $|f(x) - f_n(x)| \leq \epsilon$ for all $x \in [a,b]$ for $n \geq N$, as required.

Furthermore $f \in C([a,b])$, due to Lem. 11.10.

4.(c) Using Lem. 11.10 we see that $f' = g$ and, since $g \in C([a,b])$ (by Cor.13.2), $f \in C^1([a,b])$, as required.

5. Since

$$\left|\frac{2x}{x^2+n^3}\right| = \frac{1}{n^{3/2}}\left|\frac{2n^{3/2}x}{x^2+n^3}\right| \le \frac{1}{n^{3/2}}$$

(as $2ab \le a^2+b^2$), we have

$$\left|\arctan\frac{2x}{x^2+n^3}\right| \le \arctan\left|\frac{2x}{x^2+n^3}\right| \le \arctan n^{-3/2} \rightrightarrows 0$$

as $n\to\infty$, we see that $\arctan\frac{2x}{x^2+n^3} \rightrightarrows 0$ as $n\to\infty$.

6. Let

$$f_n(x) = \begin{cases} n^{-1} & \text{if } x\in\mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f_n is discontinuous at every point (by (Heine), as for each $x\in\mathbb{R}$ one can find a sequence of rational numbers converging to it, as well as a sequence of irrational numbers convergin to it), yet $f_n \to 0$ uniformly. (For $\epsilon > 0$ given, choose N so that $1/N < \epsilon$.)

Thm. 13.1 is not applicable here since there does not exist a single limit point x such that $\lim_{y\to x} f_n(y) = \alpha_n$, because $f_n(x)$ is discontinuous at every point.

7. Note that since f is differentiable,

$$F_n(x) := \frac{f(x+1/n)-f(x)}{1/n}$$

converges pointwise to $f'(x)$. Since f' is continuous, by the MVT, there exists some $t\in[x, x+1/n]$ such that $F_n(x) = f'(t)$.

Let $\epsilon > 0$ be given. Since f' is uniformly continuous we can choose δ such that for every x, y satisfying $|x-y|\le\delta$, $|f'(x)-f'(y)|\le\epsilon$. Then for all $n > 1/\epsilon$ (so that $1/n < \epsilon$), we have

$$|F_n(x) - f'(x)| = |f'(t) - f(x)| < \epsilon$$

for all x.

To show that the assertion fails when f' is not uniformly continuous, take $f(x) = x^3$. Then $f'(x) = 3x^2$, which is not uniformly continuous: Take $\varepsilon := 1$ and any $\delta > 0$ and note that $|f'(x) - f'(x+\delta)| = 3|2x\delta + \delta^2| > 1 = \varepsilon$ whenever x is chosen large enough. Moreover

$$\left|\frac{f(x+1/n)-f(x)}{1/n} - f'(x)\right| = \left|\frac{(x+1/n)^3 - x^3}{1/n} - 3x^2\right| = \left|\frac{3x}{n} + \frac{1}{n^2}\right| = \frac{3x}{n}.$$

for $x > 0$. Thus the left hand side cannot be bounded uniformly by a fixed $\varepsilon > 0$ for sufficiently large n (as it is not for $x := n\varepsilon$).

8. Note that

$$f(x) := \sum_{n\geq 1} (-1)^n \frac{n+x^2}{n^2} = \underbrace{\sum_{n\geq 1} \frac{(-1)^n}{n}}_{=:C} + \sum_{n\geq 1} \frac{x^2}{n^2},$$

where C is well-defined thanks to the alternating series test (Cor. 6.19). The remaining series converges uniformly on every interval $[-R,R]$, where $R>0$ (and hence on every bounded interval) due to the Weierstrass criterion (Thm 13.9), as $|x^2/n^2| \leq R^2/n^2$. Note that the last equality above is understood in the sense of limits (i.e. now that we know that both series on the right-hand side converge, $f(x)$ is well-defined and the equality holds by Cor. 7.4).

Alternatively one can use Thm. 13.8.2 to show uniform convergence on $[-R,R]$ for every $R>0$ (as $\sum_{n\geq 1}((-1)^n(n+x^2)/n^2)' = 2\sum_{n\geq 1}(-1)^n x/n^2$, which converges uniformly on $[-R,R]$ (by the Weierstrass criterion (Thm. 13.9), as $|x|\leq R$) and also we have convergence of the series in question at $x_0 := 0$).

However the given series does not converge absolutely for any fixed x since

$$\sum_{n=1}^{\infty} \frac{n+x^2}{n^2} > \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

by Ex. 6.8.

9. Let $\varepsilon > 0$ Note that, as in Thm. 6.18,

$$\left| \sum_{k=m}^{n} f_k g_k \right| = \left| \sum_{k=m}^{n-1} S_k (g_k - g_{k-1}) + S_n g_n - S_{m-1} g_m \right|$$
$$\leq M \left(\sum_{k=m}^{n-1} (g_k - g_{k-1}) + g_n + g_m \right) \leq 2M g_n \leq \varepsilon,$$

whenever n is large enough. This shows that the partial sums $\sum_{k=1}^{n} f_k g_k$ converge uniformly (to some function), by Thm. 12.12.

10. Observe that

$$f_n'(x) = \frac{1-n^2x^2}{(1+n^2x^2)^2}.$$

At $x=0$, $f_n'(0)=1$ for every n. For every $x\neq 0$ we have

$$|f_n'(x)| = \left| \frac{1/n^2 - x^2}{(1/n^2+x^2)(1+n^2x^2)} \right| \to 0$$

as $n\to\infty$. Hence the pointwise limit of $f_n'(x)$ is $f(x)$, where $f(0) := 1$, and $f(x) := 0$ for all $x\neq 0$, which is not a continuous function. Thus, since each

$f_n'(x)$ is continuous on $[-1,1]$, it cannot converge uniformly to f (as it would contradict Cor. 13.2). However, $f_n \rightrightarrows 0$ as, fixing $\varepsilon > 0$ we have

$$|f_n(x)| = \frac{1}{2n}\left|\frac{2nx}{1+n^2x^2}\right| \le \frac{1}{2n} \le \varepsilon$$

whenever $n \ge (2\varepsilon)^{-1}$.

11. Using the substitution $1-x^2 = t$,

$$\int_0^1 nx(1-x^2)^n dx = -\frac{n}{2}\int_1^0 t^n dt = \frac{n}{2}\int_0^1 t^n dt = \frac{n}{2n+2}.$$

Taking the limit as $n \to \infty$, we get that the limit equals $1/2$.

One cannot use Thm. 13.12 since $f_n(x) := nx(1-x^2)^n$ does not converge uniformly. Indeed, we verify below that $f_n(x)$ attains its maximum at $x_n := 1/\sqrt{1+2n}$, and

$$f_n(x_n) = \frac{n}{\sqrt{1+2n}}\left(1-\frac{1}{1+2n}\right)^n = \sqrt{n}\underbrace{\frac{1}{\sqrt{2+1/n}}\left(\left(1-\frac{1}{1+2n}\right)^{-(1+2n)}\cdot\frac{2n}{1+2n}\right)^{-1/2}}_{\to \frac{1}{\sqrt{2}}e^{-1/2}}$$

(where we used PS5.10) diverges to ∞ as $n \to \infty$. This contradicts uniform convergence of f_n's.

To find the maximum we note that

$$\frac{d}{dx}(nx(1-x^2)^n) = n(1-x^2)^n - 2n^2x^2(1-x^2)^{n-1},$$

which equals 0 when either $x = 1$, or $x = \pm 1/\sqrt{1+2n}$. Since $x \in [0,1]$, and since at $x = 1$, the function is 0, and since $f_n(x) \ge 0$ on $[0,1]$, each $f_n(x)$ must attain a local maximum at $x = \sqrt{1+2n}$.

Alternatively, one could have picked $x_n := 1/\sqrt{n}$ to get $f_n(x_n) = \sqrt{n}\left(1-\frac{1}{n}\right)^n \to \infty$, and deduced that there is no uniform convergence.