Math 425A Problem Sheet 2 (due 9am on Monday, 13th Sep)

Essential problems

1. (3 pts) Let X be a subset of $\mathbb{R}$. We say that $x \in X$ is *semi-isolated* if there exists a number $\varepsilon > 0$ such that at least one of the intervals $(x-\varepsilon, x)$ and $(x, x+\varepsilon)$ is disjoint with X. Prove that the set of semi-isolated points of X is at most countable. Give an example of a bounded subset of $\mathbb{R}$ with infinitely many semi-isolated points.

2. (5 pts)

(a) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$, that is for every $x \in \mathbb{R}$, $\varepsilon > 0$ there exists $q \in \mathbb{Q}$ such that $q \in (x-\varepsilon, x+\varepsilon)$. *(Hint: Use Example 1.17)*

(b) Set $k \in \mathbb{N}$. Deduce from (a) that $\mathbb{Q}^k$ is *dense* in $\mathbb{R}^k$, that is for every $x \in \mathbb{R}^k$, $\varepsilon > 0$ there exists $q \in \mathbb{Q}^k$ such that $q \in B(x,\varepsilon) := \{y \in \mathbb{R}^k : |x-y| < \varepsilon\}$ (the *open ball centered at x of radius ε*).

(c) Fix $y \in \mathbb{R}^k$ and an open ball $B(x,r) \subset \mathbb{R}^k$ containing y (i.e. such that $|x-y| < r$). Show that there exists an open ball $B(\tilde{x}, \tilde{r})$ such that $\tilde{x} \in \mathbb{Q}^k$, $\tilde{r} \in \mathbb{Q}$ and $y \in B(\tilde{x}, \tilde{r}) \subset B(x,r)$.

(d) Let E be a subset of $\mathbb{R}^k$, and let $\{B_\alpha\}_{\alpha \in A}$ be an uncountable collection of open balls $B_\alpha \subset \mathbb{R}^k$ that covers E, i.e.

$$E \subset \bigcup_{\alpha \in A} B_\alpha$$

and A is uncountable. Show that there exists a countable subcollection that still covers E, i.e. that there exists a countable subset $A_0 \subset A$ such that $E \subset \bigcup_{\alpha \in A_0} B_\alpha$. *(Hint: Consider the family of all open balls in $\mathbb{R}^k$ with centers in $\mathbb{Q}^k$ and rational radia, and use (c).)*

(Comment: Question 2(d) shows that, for ***any set*** *in $\mathbb{R}^k$, we can extract a countable subcover from any given open cover. One should compare this fact with the notion of a compact set.)*

3. (2 pts)

(a) Use the Cauchy-Schwarz inequality to show that $x + y \le \sqrt{x^2+1}\sqrt{y^2+1}$ for all $x, y \in \mathbb{R}$.

(b) Deduce from (a) that

$$\sqrt{a^2-1} + \sqrt{b^2-1} + \sqrt{c^2-1} \le \frac{ab+bc+ca}{2}$$

for all $a, b, c \ge 1$.

Additional problems

4. (1 pt) Use question 2(a) to show that the irrational numbers (i.e. $\mathbb{R} \setminus \mathbb{Q}$) are dense in $\mathbb{R}$.

5. (1 pt) Use the Cauchy-Schwarz inequality to prove the inequality between the arithmetic and quadratic means,

$$\sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \ge \frac{a_1 + a_2 + \ldots + a_n}{n} \qquad \text{for all } a_1, \ldots, a_n \ge 0.$$

When does this inequality becomes an equality?

6. (1 pt) Let $a_1, \ldots, a_n > 0$ be given. Find

$$\min \left\{ \sum_{k=1}^n a_k x_k^2 \colon x_k \in \mathbb{R} \text{ for all } k = 1, \ldots, n, \text{ and } \sum_{k=1}^n x_k = 1 \right\}.$$

7. (1 pt) Show that

$$||x| - |y|| \le |x-y| \qquad \text{for all } x, y \in \mathbb{R}^k, k \in \mathbb{N}.$$

Deduce that $|\sqrt{a^2+b^2}-\sqrt{b^2+c^2}|\leq |a-c|$ for all $a,b,c\in\mathbb{R}$.

8. (1 pt) Show that $|z+iw|^2+|w+iz|^2 = 2(|z|^2+|w|^2)$ for all $z,w\in\mathbb{C}$, and deduce that $|z+iw|^2 \leq 2(|z|^2+|w|^2)$.

9. (1 pt) Find all solutions of the equation $z^2-2z+10=0$ in $\mathbb{C}$. *(Comment: this question demonstrates an important property of the complex field $\mathbb{C}$ that every n-th degree (real or complex) polynomial has* ***exactly*** *n roots in $\mathbb{C}$ (counting multiplicity). This is known as the Fundamental Theorem of Algebra and as the fact that $\mathbb{C}$ is an* algebraically closed *field. (Recall that a real polynomial of degree n has* ***at most*** *n real roots.))*

10. (1 pt) Let $\{B_\alpha\}_{\alpha\in A}$ be any collection of pairwise disjoint open balls in R^2. Show that A is at most countable.

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