MATH 425a PS 2

1. let the set of semi-isolated point be defined as the union of:

$$\begin{cases} S_n^- := \{x \in X : (x-\frac{1}{n}, x) \cap X = \phi, \forall n>0\} \\ S_n^+ := \{x \in X : (x, x+\frac{1}{n}) \cap X = \phi, \forall n>0\} \end{cases}$$

Consider the length of 1 unit in X (also in R) $\Rightarrow$ Denote as Y.

For S_n^-, there is a maximum of $\frac{1}{1/n} = n$ semi-isolated points (countably many).

This also holds for S_n^+. Thus there are at most $2n$ semi-isolated points in Y.

Total semi-isolated points: $2n \cdot \text{card}(\mathbb{Z})$, which is countably many.

Thus $\bigcup_{n=1}^{\infty} (S_n^- \cup S_n^+)$ is countable union of countable sets, which is countable.

Example: ~~$\{x \mid x \in [0,1]\} - \{\frac{1}{2^{n-1}} \mid x \in [0,1], n \in \mathbb{Z}^+$~~ $n \in \mathbb{Z}^+\}$. $\{\frac{1}{2^{n-1}} \mid n \in \mathbb{Z}^+\}. \Rightarrow \{1, 0.5, 0.25 \dots\}$

2.

~~a) According to Archimedean property~~, given $x \in R$, $x-\varepsilon \in R$.

~~let~~ $x-\varepsilon = a$, $x+\varepsilon = b$. $\because \varepsilon > 0$, $\therefore b-a>0$. $\cancel{n(b-a)>1} \Rightarrow 1+na<nb$.

~~According to Archimedean property, $\exists n$ s.t.~~ $\exists n \in N$ s.t. ~~$nb>1 \Rightarrow n(x+\varepsilon)>1$~~

~~However~~, na must be bounded by two consecutive integers in R. $\Rightarrow c-1 \le na < c$, $c \in \mathbb{Z}$.

~~Thus we have~~ $na < c < na+1 \Rightarrow na < c < nb$, ~~$n \in N$~~ $\Rightarrow a$

a) let $x-\varepsilon = a$, $x+\varepsilon = b$. $\because \varepsilon>0$, $\therefore b-a>0$.

By A.P. $\exists n \in \mathbb{Z}^+$ s.t. $0 < \frac{1}{n} < b-a$. $\Rightarrow a < a+\frac{1}{n} < b$.

let ~~$S :=$ let $S_n :=$~~ $S = \{x \in \mathbb{Z} \mid \frac{x}{n} > a\} \Rightarrow \{x \in \mathbb{Z} \mid x > na\}$, since $n>0$.

As a non-empty set bounded below, it has a smallest element k. $k \in \mathbb{Z}$.

$\frac{k}{n} = \frac{k-1}{n} + \frac{1}{n} > a \Rightarrow \frac{k-1}{n} \le a$ because k is the smallest element of S!

$\therefore a < \frac{k}{n} = \frac{k-1}{n} + \frac{1}{n} \le a + \frac{1}{n} \Rightarrow a < \frac{k}{n} < b$. Note that $\frac{k}{n}$ is rational.

Thus, Q is dense in R as $\forall x \in R$, $\varepsilon > 0$ $\exists \frac{k}{n} \in Q$ s.t. $\frac{k}{n} \in (x-\varepsilon, x+\varepsilon)$.

b) let $x = (x_1, x_2 \dots x_n) \in R^n$. For each $1 \le m \le n$, $(x_i - \varepsilon, x_i + \varepsilon)$ includes a q_i when $\varepsilon > 0$. see (a).

Thus $\exists q = (q_1, q_2, \dots q_n)$ included in the ball of size 2ε around x.

$\therefore Q^n$ is dense in R^n.

c) let $B(x,r)$ be an open ball that contains $y=(y_1,y_2,\dots,y_n)$.

~~$\because B(x,r)$ is open $\therefore \exists B(y,\varepsilon_y) \subset B(x,r)\ \forall \varepsilon_y>0$~~

~~For $\because \mathbb{Q}^k$ dense in $\mathbb{R}^k$ $\therefore \exists q=(q_1,q_2,\dots q_n)$ contained in $B(y,\varepsilon_y)$~~

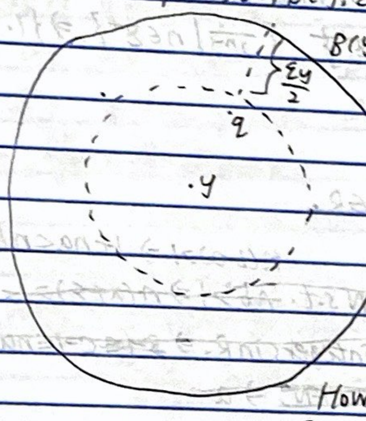
~~$\because B(y,\varepsilon_y)$ is open $\therefore$ let $n\in\mathbb{N}$ be sufficiently large s.t. $\frac{1}{n}<\varepsilon_y$~~

~~$\because B(y,\varepsilon_y)$ is open, $\exists \varepsilon_q$ s.t. $B(q,\varepsilon_q)\subset B(y,\varepsilon_y)$~~

~~Now pick $n\in\mathbb{N}$ sufficiently large s.t. $\frac{1}{N}<\varepsilon_q$~~

~~$\Rightarrow B(q,\frac{1}{n})\subset B(q,\varepsilon_q)\subset B(y,\varepsilon_y)\subset B(x,r)$~~

$\because B(x,r)$ is open $\therefore \exists B(y,\varepsilon_y)\subset B(x,r)\ \forall \varepsilon_y>0$.



Construct a ball with radius $\frac{\varepsilon_y}{2}$. $(q_1,q_2,\dots q_n)$

$\because \mathbb{Q}^k$ dense in $\mathbb{R}^k$, we find a point q in that ball.

~~$\because$~~ radius of q must $\le \frac{\varepsilon_y}{2}$ or it will be outside $B(y,\varepsilon_y)$

Also, if radius of $q \ge \frac{\varepsilon}{2}$, it will include y.

Thus we have a $B(q,\frac{\varepsilon_y}{2})$ that will include y.

↳ $\subset B(y,\varepsilon_y)$.

However, we need to let the radius to be rational.

We know radius of $q \le \frac{\varepsilon_y}{2}$. We need its radius $\ge \|y-q\|$ to make sure y is included.

$\because \mathbb{Q}$ dense in $\mathbb{R}$, $\therefore$ there is some rational number $\|y-q\| < \underset{\tilde{r}}{p} < \frac{\varepsilon_y}{2}$ that satisfies this.

$\therefore$ We have $y\in B(\tilde{q},\tilde{r})\subset B(y,\varepsilon_y)\subset B(x,r)$.

↳ Denote q by $\tilde{q}\in\mathbb{Q}$

d) For the family of all open balls $\mathbb{R}^k$ with centers in $\mathbb{Q}^k$ and rational radii,

~~they~~ there are countably many of them since $\mathbb{Q}^k$ and $\mathbb{Q}$ are countable. $B(q,r)$ covers E ($\mathbb{Q}^k$ dense in $\mathbb{R}^k$).

~~* Pick $x\in E$. $x\in B(q,r)\subset B(\alpha)$ by (c).~~ $\forall x\in E$, $\exists B_\alpha$ s.t. $x\in B(q,r)\subset B_\alpha$. by 2c)

$\because \bigcup_{q\in\mathbb{Q}^k, r\in\mathbb{Q}} B(q,r)$ countable, $\therefore B_\alpha$ must also be countable. Otherwise $B(q,r)$ uncountable.

$\therefore B_\alpha$ countable. $\therefore \exists A_0\subset A$ s.t. $\bigcup_{\alpha\in A_0} B_\alpha \supset E$.

3.

a) Let two vectors be $\langle 1, x\rangle, \langle y, 1\rangle$.

$\langle 1,x\rangle \cdot \langle y,1\rangle \leq |\langle 1,x\rangle|\,|\langle y,1\rangle|$

$\therefore\ x+y \leq \sqrt{x^2+1}\cdot\sqrt{y^2+1}$

b) $\sqrt{a^2-1}+\sqrt{b^2-1} < \sqrt{a^2}+\sqrt{b^2} \leq ab.$

$\hookrightarrow \sqrt{b^2-1}+\sqrt{c^2-1} < bc,\ \sqrt{a^2-1}+\sqrt{c^2-1} < ac.$

$\therefore\ \frac{2(\sqrt{a^2-1}+\sqrt{b^2-1}+\sqrt{c^2-1})}{2} < \cancel{\frac{ab+bc}{2}}\ \frac{ab+bc+ac}{2}$

$\hookrightarrow \sqrt{a^2-1}+\sqrt{b^2-1}+\sqrt{c^2-1} < \frac{ab+bc+ac}{2}.$

4. We know Q is dense in R. pick a random irrational number, $\sqrt{2}$.

$\because Q+\sqrt{2}$ irrational, $\underbrace{Q+\sqrt{2}}_{\text{subset of } R\setminus Q}$ dense in $\underbrace{R+\sqrt{2}}_{R}$ $\Rightarrow$ irrationals dense in R.

5. let $\frac{a_1+a_2+\cdots+a_n}{n} = a_1(\frac{1}{n})+a_2(\frac{1}{n})+\cdots+a_3(\frac{1}{n})$

then $(\frac{a_1+a_2+\cdots+a_n}{n})^2 \leq (a_1^2+a_2^2+\cdots+a_n^2)\underbrace{((\frac{1}{n})^2+\cdots+(\frac{1}{n})^2)}_{n \text{ terms}}$

$\leq \frac{a_1^2+\cdots+a_n^2}{n}$

$\therefore\ \frac{a_1+\cdots+a_n}{n} \leq \sqrt{\frac{a_1^2+\cdots+a_n^2}{n}}$ The equality holds when $\cancel{a_1=a_2=\cdots=a_n=a}$, $(\frac{a_1}{n})=(\frac{a_2}{n})=\cdots(\frac{a_n}{n})$ or $a_1=a_2=\cdots=a_n$

6. By C-S, $(\sum x_k)^2 = (\sum \sqrt{a_k}\, x_k \frac{1}{\sqrt{a_k}})^2 \leq \sum a_k x_k^2 \sum \frac{1}{a_k}$

$\hookrightarrow \sum a_k x_k^2 \geq \frac{1}{\sum \frac{1}{a_k}}$. To minimize, $\frac{a_k x_k^2}{\cancel{x}\,1/a_k}$ must be constant $\Rightarrow a_k^2 x_k^2$ constant, $x_k \propto \frac{1}{a_k}$.

let $x_k = \frac{c}{a_k}$. $c = \cancel{\sum \frac{1}{1/a_k}}\ \frac{1}{\sum a_k}$ so that $\sum x_k = \cancel{\sum \frac{1}{1/a_k}\cdot\sum\frac{1}{a_k} = 1}\ \frac{1}{\sum a_k}\cdot\sum\frac{1}{a_k} = 1.$

$\therefore \min \sum a_k x_k^2 = \sum a_k \cdot \frac{c^2}{a_k^2} = (\sum \frac{1}{1/a_k})^{-2}\sum\frac{1}{a_k} = \cancel{\sum\frac{1}{1/a_k}}\cdot\frac{1}{\sum a_k}$

7. ~~let $x>y>0$. $||x|-|y|| = |x-y|$.~~ ~~let $x>0>y$. $||x|-|y|| = x-y$~~

~~let $0>x>y$. $||x|-|y|| = x-y \neq -|x-y|$~~

~~let $y>x>0$.~~

$|y| = |x+(y-x)| \leq |x|+|y-x| = |x|+|x-y| \Rightarrow |y|-|x| \leq |x-y|$, $|x|-|y| \geq \cancel{|x-y|}\ -|x-y|$.

~~According to absolute value properties.~~ $\because |x| = |x-y+y| \leq |x-y|+|y|$, $\therefore |x|-|y| \leq |x-y|$.

$\Rightarrow -|x-y| \leq |x|-|y| \leq |x-y|$, Hence $||x|-|y|| \leq |x-y|$.

let x be (b,a), y be (b,c) $\Rightarrow$ Special case in R^2.

$||x|-|y|| = |\sqrt{b^2+a^2}-\sqrt{b^2+c^2}| \leq |(b,a)-(b,c)| = |a-c|.$

8. $|z+iw|^2 \cancel{+|w+iz|^2 = (\bar{z}+i\bar{w})(\bar{z}-iw)+(\bar{w}+iz)(w-iz)} = (z+iw)(\bar{z}-\overline{iw})$

$\cancel{= z^2+w^2}$ $= z\bar{z} + iw\bar{z} - iz\bar{w} + w\bar{w} = |z|^2+|w|^2 + i(w\bar{z} - \cancel{z}z\bar{w})$

$|w+iz|^2 = (w+iz)(\bar{w}\cancel{+}\overline{iz}) = w\bar{w} + iz\bar{w} - iw\bar{z} + z\bar{z} = |z|^2+|w|^2 + i(z\bar{w} - w\bar{z})$

$\therefore |z+iw|^2 + |w+iz|^2 = 2(|z|^2+|w|^2)$

$\because |w+iz|^2 \geq 0, \therefore |z+iw|^2 \leq 2(|z|^2+|w|^2)$.

9. $z = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2\pm 6i}{2}$. $z_1 = 1+3i$, $z_2 = 1-3i$.

10. Note that Q^2 is dense in R^2. let E denote the set covered by $B\alpha$.

Consider ~~a~~ the family of open balls with centers $^{(q)}$ in Q^2 and radii $^{(r)}$ ~~Ba~~ rational $\Rightarrow$ countable.

$\forall x \in E$, $\exists x \in B(q,r) \subset \cancel{B\epsilon}\, B\alpha$. $\Rightarrow B(q,r)$ countable.

A must be countable as $\{\cancel{B\alpha}\, B_\alpha\}_{\alpha\in A}$ countable $\Rightarrow$ otherwise $B(q,r)$ not countable.

($\cancel{z} \rightarrow Q$ injection).

$\cancel{\{B\alpha\}}$

$\{B\alpha\}_{\alpha\in A}$