Solutions to Problem Set 2

Math 425a, Spring 2021

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1. Let S be the set of all semi-isolated points of X. We will first divide S into two parts as follows-

$$E_R = \left\{ x \in S \middle| \exists\, \varepsilon \text{ such that } (x, x+\varepsilon) \cap X = \phi \right\}$$

$$E_L = \left\{ x \in S \middle| \exists\, \varepsilon \text{ such that } (x-\varepsilon, x) \cap X = \phi \right\}$$

Clearly, $S = E_R \cup E_L$. Therefore it is enough to show that these sets are countable, because then S being an union of two countable sets will again be countable. We shall prove E_R is countable, the proof for E_L is exactly the same (modulo a few change of signs).

Proposition. E_R is a countable set

Proof. By definition, given $x \in E_R$ we can find ε_x such that $(x, x+\varepsilon_x) \cap X = \phi$. Take another $y \neq x$ in E_R and similarly find ε_y such that $(y, y+\varepsilon_y) \cap X = \phi$. We claim that these two intervals $(x, x+\varepsilon_x)$ and $(y, y+\varepsilon_y)$ are disjoint. Indeed since $(x, x+\varepsilon_x) \cap X = \phi$, we can conclude $y \notin (x, x+\varepsilon_x)$ which implies either $y \leq x$ or $y \geq (x+\varepsilon_x)$. If $y \geq (x+\varepsilon_x)$ then obviously $(x, x+\varepsilon_x) \cap (y, y+\varepsilon_y) = \phi$. On the other hand, if $y \leq x$ then the fact that $x \notin (y, y+\varepsilon_y)$ forces x to be bigger than $(y+\varepsilon_y)$ guaranteeing $(x, x+\varepsilon_x) \cap (y, y+\varepsilon_y) = \phi$ in this case too.

Now for each $x \in E_R$ construct such pairwise disjoint intervals and assign to each x a rational q_x in that interval. This assignment is clearly $1-1$ as the intervals are pairwise disjoint. Therefore we have established a bijection from E_R to some subset of $\mathbb{Q}$ and we can conclude that E_R is countable.

$\square$

2. (a) It is equivalent to show that for any $a < b$ in $\mathbb{R}$, we can find a rational q such that $a < q < b$. To that end, by the Archimedean property, we can choose $n \in \mathbb{N}$ such that $n(b-a) > 1$. Next choose $m \in \mathbb{N}$ such that $m \leq an < m+1$. Such a m exists because the set $\left\{ p \in \mathbb{N} \;\middle|\; p \leq an \right\}$ is a bounded set of natural numbers and is

therefore guaranteed to have a maximum. Now we have the following string of inequalities

$$na < m + 1 \le na + 1 < nb$$

implying, $a < (m + 1)/n < b$. As $(m + 1)/n$ is clearly rational we are done.

(b) We have to show that for any $\mathbf{x} \in \mathbb{R}^k$ and $\varepsilon > 0$ there exists a $\mathbf{q} \in \mathbb{Q}^k$ such that $|\mathbf{x} - \mathbf{q}| < \varepsilon$. Let $\mathbf{x} = (x_1, x_2, \cdots, x_k)$. For each x_i, due to part (a), we can choose $q_i \in \mathbb{Q}$ such that $|x_i - q_i| < \frac{\varepsilon}{\sqrt{k}}$. Define $\mathbf{q} = (q_1, q_2, \cdots, q_k)$ with the q_i's that we have chosen. Then $|\mathbf{x} - \mathbf{q}| = \sqrt{(x_1 - q_1)^2 + \cdots + (x_k - q_k)^2} < \varepsilon$, as desired.

(c) Let $y \in R^k$ and $B(x, r) \subset \mathbb{R}^k$ be as given in the problem. Since $B(x, r)$ is an open subset, by definition there exists some r' such that $B(y, r') \subset B(x, r)$. Choose some rational point $q \in \mathbb{Q}^k$, such that $|y - q| < \frac{r'}{3}$ (this can be done due to part (b)). Also choose some rational radius r'' such that $\frac{r'}{3} < r'' < \frac{2r'}{3}$. Consider $B(q, r'')$, clearly $y \in B(q, r'')$ as $|y - q| < \frac{r'}{3} < r''$. Next, for any $w \in B(q, r'')$, we have that $|y - w| < |y - q| + |q - w| < \frac{r'}{3} + r'' < \frac{r'}{3} + \frac{2r'}{3} = r'$, implying $w \in B(y, r')$. Therefore we have that $B(q, r'') \subset B(y, r')$ and it contains y. Since, $B(y, r') \subset B(x, r)$ by construction, the assertion follows.

(d) Let E and $\{B_\alpha\}_{\alpha \in A}$ be as specified. Also let U to be the collection of open balls in $\mathbb{R}^k$ that have rational center and rational radius. Clearly U is a countable set (each such ball is uniquely specified by it's center and radius, and the set of all such centers and radius is $\mathbb{Q}^k \times \mathbb{Q}$ which is the same as $\mathbb{Q}^{k+1}$ and hence countable). Now for each $e \in E$ we have a B_α such that $e \in B_\alpha$ (because it's a cover). From part (c), we have some element $U_\lambda \in U$ such that $e \in U_\lambda \subset B_\alpha$. Therefore the set of such U_λ's also cover E. This is a countable set, because U is countable but we're not done yet because this is a different open cover than the one given to us and we need to find a sub-cover of the original. To get that, now **choose one** $B_\lambda \supset U_\lambda$ for each U_λ that covers E. Since U_λ's were countable this new set of B_λ's is countable and forms the desired sub-cover.

3. (a) Apply Cauchy Schwarz inequality with $n = 2$, $a_1 = 1, a_2 = y$ and $b_1 = x, b_2 = 1$.

(b) Let $x = \sqrt{a^2 - 1}$ and $y = \sqrt{b^2 - 1}$. Applying part (a) yields

$$\sqrt{a^2 - 1} + \sqrt{b^2 - 1} \le ab$$

Similarly we can get

$$\sqrt{b^2 - 1} + \sqrt{c^2 - 1} \le bc$$

and

$$\sqrt{c^2 - 1} + \sqrt{a^2 - 1} \le ca$$

Adding all these gives the desired inequality.

4. We have to exhibit that for any $a < b$ in $\mathbb{R}$ there exist a $p \in \mathbb{Q}^c$ such that $a < p < b$. Consider $\frac{a}{\sqrt{2}}$ and $\frac{b}{\sqrt{2}}$. By 2(a), there exists $q \in \mathbb{Q}$ such that

$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}}$$

or

$$a < \sqrt{2}q < b$$

Since $q \in \mathbb{Q}$, it follows that $\sqrt{2}q \in \mathbb{Q}^c$ and hence the result.

5. Apply Cauchy Schwarz with $a_1, a_2, \cdots, a_n$ as given and $b_1 = b_2 = \cdots = b_n = 1$ to get the inequality, to obtain

$$a_1 + \ldots + a_n = (a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) \leq |(a_1, \ldots, a_n)| \cdot |(b_1, \ldots, b_n)| = \sqrt{a_1^2 + \ldots + a_n^2} \cdot \sqrt{n},$$

as required.

Equality holds in Cauchy Schwarz if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$, therefore in this case it holds if all the a_i's are equal.

6. Let $p_k = \frac{1}{\sqrt{a_k}}$ and $q_k = \sqrt{a_k} x_k$ for all $k = 1, 2, \cdots, n$. Note that since all a_k's are given to be positive, taking square roots is justified. Applying Cauchy-Schwarz to the set of p_k's and q_k's yields

$$\left(\sum_{k=1}^{k=n} p_k q_k \right)^2 \leq \left(\sum_{k=1}^{k=n} p_k^2 \right) \left(\sum_{k=1}^{k=n} q_k^2 \right)$$

Putting in the values for p_k and q_k we have,

$$\left(\sum_{k=1}^{k=n} x_k \right)^2 \leq \left(\sum_{k=1}^{k=n} a_k x_k^2 \right) \left(\sum_{k=1}^{k=n} \frac{1}{a_k} \right)$$

Since, $\sum_{k=1}^{k=n} x_k = 1$, we have that

$$\sum_{k=1}^{k=n} a_k x_k^2 \geq \left(\frac{1}{\sum_{k=1}^{k=n} \frac{1}{a_k}} \right)$$

which gives us the minimum value as $\left(\frac{1}{\sum_{k=1}^{k=n} \frac{1}{a_k}} \right)$. Note here that since the equality is achieved in certain cases in Cauchy Schwarz, the minimum here is also attained in those cases.

7. (a)

$$|x| = |x - y + y| \leq |x - y| + |y|$$

and

$$|y| = |y - x + x| \leq |y - x| + |x| = |x - y| + |x|$$

Therefore we have,

$$|x| - |y| \leq |x - y|$$
$$|y| - |x| \leq |x - y|$$

or,

$$|x| - |y| \leq |x - y|$$
$$|x| - |y| \geq -|x - y|$$

Combining these two gives us the required inequality.

(b) Set $\mathbf{x} = (a, b)$ and $\mathbf{y} = (c, b)$ in $\mathbb{R}^2$ and apply the inequality from part (a).

8. Recall that for any $a \in \mathbb{C}$, we have that $|a|^2 = a\bar{a}$. Therefore (by Thm. 2.12),

$$\begin{aligned}|z + iw|^2 &= (z + iw)\overline{(z + iw)} \\ &= (z + iw)(\bar{z} - i\bar{w}) \\ &= |z|^2 + |w|^2 + iw\bar{z} - iz\bar{w}\end{aligned}$$

Similarly,

$$\begin{aligned}|w + iz|^2 &= (w + iz)\overline{(w + iz)} \\ &= (w + iz)(\bar{w} - i\bar{z}) \\ &= |w|^2 + |z|^2 + iz\bar{w} - iw\bar{z}\end{aligned}$$

Adding, we get the desired identity-

$$|z + iw|^2 + |w + iz|^2 = 2(|z|^2 + |w|^2)$$

Since $|w + iz|^2$ is always non negative, the second conclusion follows.

9.

$$\begin{aligned}z^2 - 2z + 10 &= 0 \\ (z - 1)^2 + 9 &= 0 \\ (z - 1) &= \pm 3i\end{aligned}$$

This shows that $(1 + 3i)$ and $(1 - 3i)$ are roots. In order to show that these are the only roots note that

$$z^2 - 2z + 10 = (z - (1 + 3i))(z - (1 - 3i))$$

and so by the field property of $\mathbb{C}$ (Lem. 1.12(iii)) the equation $(z-(1+3i))(z-(1-3i)) = 0$ implies that either $z = 1 + 3i$ or $z = 1 - 3i$. Hence these are the only roots.

10. Recall that $\mathbb{Q}^2 \subset \mathbb{R}^2$ is dense in $\mathbb{R}^2$ (by question 2(b)) and countable too (by Ex. 2.20.3). Associate to each ball some point in $\mathbb{Q}^2$ that lives inside it (such a point exists because of density). This association is clearly $1 - 1$ as the balls are pairwise disjoint. Therefore we have established a bijection from the collection of balls given to us to some subset of $\mathbb{Q}^2$.