Math 425A Problem Sheet 3 (due 9am on Wednesday, 22nd Sep)

Essential problems

1. (3 pts) Let $X := \mathbb{R}^2$. We say that $x, y \in X$ *lie on the same ray* if

$$y = 0, x = 0 \text{ or } y = \lambda x \text{ for some } \lambda > 0.$$

Given $x, y \in X$ we set

$$\tilde{d}(x,y) := \begin{cases} |x-y| & \text{if } x, y \text{ lie on the same ray,} \\ |x| + |y| & \text{if not.} \end{cases}$$

(a) Show that $(X, \tilde{d})$ is a metric space.

(b) Let $d(x,y) := |x-y|$ denote the ("usual") Euclidean metric on X. Show that convergence with respect to $\tilde{d}$ implies convergence with respect to d (i.e. if $(x_n)_{n\geq 1} \subset X$ is any sequence such that $\tilde{d}(x_n, x) \to 0$ for some $x \in X$ then $d(x_n, x) \to 0$), but that the opposite implication is false (i.e. find a sequence of points $(x_n, y_n) \in \mathbb{R}^2$, $n \geq 1$, and $(x, y) \in \mathbb{R}^2$ such that $(x_n - x)^2 + (y_n - y)^2 \to 0$, but $\tilde{d}((x_n, y_n), (x, y)) \not\to 0$, as $n \to \infty$).

2. (4 pts) Let (X, d) be a metric space. Let A, B be disjoint, nonempty, compact subsets of X. Show that there exist points $a_0 \in A$, $b_0 \in B$ that are the closest together, that is

$$d(a_0, b_0) \leq d(a, b) \qquad \text{for all } a \in A, b \in B.$$

(Hint: Consider $\inf\{d(a,b) \colon a \in A, b \in B\}$*.)*

3. (3 pts) Let $(X, d) := (\mathbb{R}, |\cdot|)$.

(a) Suppose that a sequence $(x_n)_{n\geq 1} \subset X$ converges to some $x \in X$, i.e. $x_n \to x$ as $n \to \infty$. Show that then the sequence of averages of the first n elements converges to x as well, i.e.

$$\frac{1}{n}(x_1 + x_2 + \ldots x_n) \to x \qquad \text{as } n \to \infty. \tag{1}$$

(b) Show that the opposite implication is false; namely give an example of a sequence $(x_n)_{n\geq 1} \subset \mathbb{R}$ such that (1) holds for some $x \in \mathbb{R}$, but $x_n \not\to x$ as $n \to \infty$.

Additional problems

4. (1 pt) For $x, y \in \mathbb{R}$ let $d(x,y) := |\sin|x-y||$. Is d a metric on $\mathbb{R}$?

5. (1 pt) Let $X := \mathbb{R}^2$. Given $p \in [1, \infty)$ let

$$d_p(x,y) := \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{\frac{1}{p}},$$

and

$$d_\infty(x,y) := \max(|x_1 - y_1|, |x_2 - y_2|),$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. You can assume that $(\mathbb{R}^2, d_p)$ is a metric space for every $p \in [1, \infty]$.

(a) Sketch, in a single figure, the unit neighbourhood $N_1(0)$ using the metric d_p with different values of the parameter p (say for $p = 1$, some $p \in (1, 2)$, $p = 2$, some $p \in (2, \infty)$, and $p = \infty$).

(b) Show that $d_p(x, y) \leq c\, d_q(x, y)$ for $p, q \in [1, \infty]$, $x, y \in \mathbb{R}^2$, where $c > 0$ is independent of p, q, x, y.

(Comment: these are the so-called "l^p metrics" (or "l^p norms") on $\mathbb{R}^2$. Property (b) shows they are all equivalent to each other (namely that convergence with respect to one of these metric implies convergence with respect to all others), which is an important property of $\mathbb{R}^2$ (and, by the way, it is also true in $\mathbb{R}^k$ for any $k \in \mathbb{N}$), but is often false in general metric spaces (for example in many infinite dimensional spaces, see also question 1 for a simple example).)

6. (1 pt) Let $n \in \mathbb{N}$

$$P_n \coloneqq \{p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0 \colon a_k \in \mathbb{R} \text{ for } k = 1, \ldots, n \text{ and } a_n \neq 0\}$$

be the set of all real polynomials of order n, and let $d \colon P_n \times P_n \to \mathbb{R}$ be defined by

$$d(p_n, q_n) \coloneqq |a_n - b_n| + \ldots + |a_0 - b_0|,$$

where $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$, $q_n(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots b_1 x + b_0$.

(a) Show that d is a metric on P_n.

(b) Now fix $k \in \{0, \ldots, n\}$ and $\tilde{d}(p_n, q_n) \coloneqq |a_k - b_k|$. Is $\tilde{d}$ a metric on P_n?

7. (1 pt) Let (X, d) be a metric space. Show that $\emptyset$ (the empty set) and X are both closed and open.

8. (1 pt) Let $(X, d) \coloneqq (\mathbb{R}, |\cdot|)$ and suppose that $x_n \to x$ as $n \to \infty$. Show that then $|x_n| \to |x|$ and $(x_n - 1)^2 \to (x - 1)^2$ as $n \to \infty$.

9. (1 pt) Prove Theorem 3.15 using Theorem 3.16.

10. (1 pt) Let $(X, d) \coloneqq (\mathbb{R}, |\cdot|)$ and

$$A \coloneqq \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \subset X.$$

Show that A is compact.

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