MATH 425a problem set 3

1.

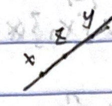
a) First, suppose x, y lie on the same ray. $\tilde{d}(x,y) = |x-y|$.

1) $|x-y| \geq 0$ ✓, $d=0$ only when $x=y$ $\cancel{\Leftrightarrow}$ $(y = \lambda x,\ \lambda = 1)$. ✓

2) $|x-y| = |y-x|$, $|x-\lambda x| = |\lambda x - x|$ ✓.

3) $\tilde{d}(x,y) \leq \tilde{d}(x,z) + \tilde{d}(z,y)$.

i) x, y, z on the same ray. $|x-y| \leq |x-z| + |z-y|$ ✓.



ii) $\cancel{\neq}$ z not on the same ray as x, y. $|x-y| \leq |x| + |z| + |z| + |y|$ ✓.

Now suppose x, y not on the same ray. $\tilde{d}(x,y) = |x| + |y|$.

1) $|x| + |y| \geq 0$ ✓.

2) ~~$|x|$~~ $|x| + |y| = |y| + |x|$ ✓.

3) $\tilde{d}(x,y) \leq \tilde{d}(x,z) + \tilde{d}(z,y)$.

i) z on the same ray with either x, y. $|x| + |y| \leq |x-z| + |y| + |z|$ ✓

ii) z not on the same ray with x or y. $|x| + |y| \leq |x| + |z| + |z| + |y|$ ✓.

Thus we see ~~X~~ $(X, \tilde{d})$ is a metric space.

b) Proof: Note that ~~$|x|$~~ $|x-y| = |x+(-y)| \leq |x| + |-y| = |x| + |y|$.

Thus, $d(x,y) = |x-y| \leq \tilde{d}(x,y)$ always true. Hence, if

$\tilde{d}(x,y) \to 0$, ~~x~~ $d(x,y)$ ~~will alwa~~ $\to 0$ as well. opposite: Consider $\{1, \frac{1}{n}\}_{n\geq 1}$.

2.

Consider $\{(a_n, b_n)\}_{n\geq 1}$ that converges to $\inf\{d(a,b) : a \in A, b \in B\}$.

$\because A, B$ compact, $\therefore \{a_n\}_{n\geq 1}$ admits a subsequence $\{x_{n_k}\}$ that converges to $x \in A$,

~~$\{b_n\}$~~ $\{y_{n_k}\}$ admits another subsequence that converges to $y \in B$.

Thus $d(x,y) \leq d(x - x_j) + d(x_{n_k} - y) \leq d(x - x_{n_k}) + d(x_{n_k} - y_{n_k}) + d(y_{n_k} - y)$ for all j.

Letting $n_k \to \infty$, $d(x,y) \leq \lim_{n_k \to \infty} d(x_{n_k}, y_{n_k})$ (as $\lim_{n_k \to \infty} d(x - x_{n_k}) = \lim_{n_k \to \infty} (y_{n_k} - y) = 0$).

Hence $d(x,y) = \inf\{d(a,b)\}$.

↳ $d(a_0, b_0) \leq d(a,b)$ follows.

~~Opposite example: $\{1, \frac{1}{n}\}_{n\geq 1}$.~~

3. We want to show $\left|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right| < \varepsilon$ for some n $(\varepsilon > 0)$.

a) $\left|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right| \le \underbrace{\left|\frac{1}{n}\sum_{i=1}^{m}(x_i - x)\right|}_{①} + \underbrace{\left|\frac{1}{n}\sum_{i=m+1}^{n}(x_i - x)\right|}_{②}$, $m<n$.

for ①, $\left|\frac{1}{n}\sum_{i=1}^{m}(x_i-x)\right| \le \frac{1}{n}\sum_{i=1}^{m}|x_i - x| < \frac{mM}{n}$.

↳ $\because \{x_n\} \to x$, $\therefore \exists \underset{>0}{M}$ s.t. $|x_i - x| < M$.

also, $|x_i - x| < \frac{\varepsilon}{2}$.

for ②, $\left|\frac{1}{n}\sum_{i=m+1}^{n}(x_i - x)\right| \le \frac{1}{n}\sum_{i=m+1}^{n}|x_i - x| < \frac{n-m-1}{n}\cdot\frac{\varepsilon}{2} < \frac{\varepsilon}{2}$ $\left(0 < \frac{n-m-1}{n} < 1\right)$, $\varepsilon > 0$.

~~We must~~

Now we pick n large enough s.t. $\frac{mM}{n} < \frac{\varepsilon}{2}$, i.e. $n > \frac{2mM}{\varepsilon}$.

Thus $① < \frac{mM}{n} < \frac{\varepsilon}{2}$, $② < \frac{\varepsilon}{2}$, $①+② < \varepsilon \Rightarrow$ fitting the limit definition.

Hence $\left|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right| < \varepsilon$ for $\varepsilon > 0$, $\frac{1}{n}\sum_{i=1}^{n} x_i \to x$ as $n \to \infty$.

b) Consider $\{(-1)^n\}_{n\ge1}$. $\frac{1}{n}\sum_{i=1}^{n} x_i \to 0$, but $x_n \not\to 0$ as $n \to \infty$.

4.

No. Consider $x = 2\pi$, $y = \pi$, $d(x,y) = 0$.

5.

6. (see next page).

7.

ϕ is open. By definition, $\forall x \in E$, $\exists N_\varepsilon(x)$, $\varepsilon > 0$ s.t. $N_\varepsilon(x) \subset E$.

↳ There are no points in ϕ. So this is automatically satisfied.

Likewise, ϕ is closed because there are no points (no limit points) in it as well.

The complement of ϕ, which is X, must be both open and closed.

8.

9.

Proof: Suppose K is compact. Then ~~∃~~ every sequence in K admits a convergent subsequence, i.e. $\forall (x_n)_{n\geq 1} \subset K$, $\exists (x_{n_k})_{k\geq 1}$ that converges to $x \in K$. Thus x is a the limit point for every sequence $x_{n_k} \subset K$.
Therefore K contains all its limit point, so K is compact $\Rightarrow$ it's closed.

10.

A is bounded. ~~for $M \geq 1$,~~ This is because $0 \leq \{A\} \leq 1$.
A is closed. 0 is the only limit point as ~~$\exists \varepsilon > 0$~~ $\forall \varepsilon > 0$, $\exists N_\varepsilon(0)$ that contains another point $q \in A$, $q \neq 0$.
According to Heine–Borel theorem A is compact.

6.

a) 1) $d(p_n, q_n) = |a_n - b_n| + \cdots + |a_0 - b_0| \geq 0$ ✓, $d(p_n, q_n) = 0$ when $p_n = q_n$ ✓.

2) ~~$d(p_n, q_n) =$~~ $|a_n - b_n| + \cdots + |a_0 - b_0| = |b_n - a_n| + \cdots + |b_0 - a_0| \Rightarrow d(p_n, q_n) = d(q_n, p_n)$.

3) $d(p_n, q_n) \leq d(p_n, r_n) + d(r_n, q_n)$ True or not?

$\hookrightarrow |a_n - \overset{c_n}{\cancel{r_n}}| + \cdots + |a_0 - \overset{c}{\cancel{r_0}}| + |\overset{c_n - b_n}{\cancel{r_n - q_n}}| + \cdots + |\overset{c_0 - b_0}{\cancel{r_0 - q_0}}|$

$|a_n - b_n| =$

$|a_n - c_n| + |c_n - b_n| \geq |a_n - b_n|$ due to triangular inequality. $(|a_n - c_n + c_n - b_n|)$

$\therefore |a_n - c_n| + \cdots + |a_0 - c_0| + |c_n - b_n| + \cdots + |c_0 - b_0| \geq |a_n - b_n| + \cdots + |a_0 - b_0|$. 3) is true.

Thus, d is a metric space on P_n.

b) No. when $d'(p_n, q_n) = 0$, p_n does not have to be equal to q_n. Other coefficients apart from the pair (a_k, b_k) can be arbitrary.