Homework Assignment Solution 3

September 25, 2021

#1 (a)

Clearly $\widetilde{d}$ is non-negative, and when $x = y = \mathbf{0}$, since they lie on the same ray, $\widetilde{d}(\mathbf{0},\mathbf{0}) = 0$. Assume now $\widetilde{d}(x,y) = 0$. In the first case we require $x - y = \mathbf{0} \iff x = y$. In the second case, $|x| = |y| = 0 \iff x = y = \mathbf{0}$.

It is evidently symmetric, so we will finish the proof by showing that $\widetilde{d}$ satisfies the triangle inequality. Let $x,y,z \in \mathbf{R}^2$ be given. If x and z lie on the same ray, then there are 3 possibilities: (i) both the pairs x,y and y,z lie on the same ray; (ii) only one of the pairs lie on the same ray (say, x,y without loss of generality); and (iii) none of the pairs x,y and y,z lie on he same ray.

In the case (i),

$$\widetilde{d}(x,y) + \widetilde{d}(y,z) = |x-y| + |y-z| \geq |x-y+y-z| = |x-z| = \widetilde{d}(x,z)$$

by the triangle inequality for $|\cdot|$. In the case (ii),

$$\widetilde{d}(x,y) + \widetilde{d}(y,z) = |x| + |y| + |y-z| \geq |x-y| + |y-z| \geq |x-z| = \widetilde{d}(x,z).$$

We can see without much difficulty that when y,z also lie on the same ray, the inequality also holds.

Observe that the relation of lying on the same ray is a transitive relation, meaning that if x,y and y,z lie on the same ray, then so do x,z. This is because if $y = \lambda_1 x$ and $z = \lambda_2 y$, then $z = (\lambda_1\lambda_2)x$, or $x = y = z = 0$. Hence under the assumption that x,z don't lie on the same ray, (i) cannot occur. In (ii),

$$\widetilde{d}(x,y) + \widetilde{d}(y,z) = |x| + |y| + |z-y| \geq |x| + |y+z-y| = |x| + |z| = \widetilde{d}(x,z).$$

In the case (iii),

$$\widetilde{d}(x,y) + \widetilde{d}(y,z) = |x| + |y| + |y| + |z| \geq |x| + |z| = \widetilde{d}(x,z),$$

as $|y| \geq 0$.

#1 (b)

Observe that $\widetilde{d}(x,y) \geq d(x,y)$ for every pair $(x,y) \in \mathbb{R}^2$. Hence if $\widetilde{d}(x_n,x) \to 0$, then for any $\epsilon > 0$, there exists N such that for all $n > N$, $\epsilon > \widetilde{d}(x_n,x) > d(x_n,x) \geq 0$. Hence if $x_n \to x$ in $\widetilde{d}$, then $x_n \to x$ in d as well.

Consider $(x_n) = (n^{-1}, 1)$. This converges to $x := (0, 1)$ in d, but since none of the points in the sequence lies on the say ray with $(0, 1)$ (as the x-coordinate is non-zero), $\widetilde{d}(x_n, x) = |x_n| + |x| = \sqrt{n^{-2} + 1} + 1 \to 2$ as $n \to \infty$.

#2

We first prove the "reverse triangle inequality" for metric: For every $x, y, z \in (X, d)$, $|d(x, y) - d(y, z)| \leq d(x, z)$. (This is a generalisation of the claim from PS2.7 to any metric.)

Proof. Let $x, y, z \in X$ be given. Then we have

$$d(x, y) \leq d(x, z) + d(z, y) = d(x, z) + d(y, z) \implies d(x, y) - d(y, z) \leq d(x, z),$$

and

$$d(y, z) \leq d(y, x) + d(x, z) = d(x, y) + d(x, z) \implies -d(x, z) \leq d(x, y) - d(y, z).$$

Combining these two inequalities, we obtain

$$|d(x, y) - d(y, z)| \leq d(x, z). \qquad \blacksquare$$

Next, we prove that if $(x_n) \to x$ and $(y_n) \to y$ in d, then $d(x_n, y_n) \to d(x, y)$.

Proof. Let $\epsilon > 0$ be given. By the triangle inequality,

$$\begin{aligned} |d(x_n, y_n) - d(x, y)| &= |d(x_n, y_n) - d(x_n, y) + d(x_n, y) - d(x, y)| \\ &\leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \\ &= |d(y_n, x_n) - d(x_n, y)| + |d(x_n, y) - d(y, x)| \\ &\leq d(y_n, y) + d(x_n, x). \end{aligned}$$

Because $x_n \to x$ and $y_n \to y$, there exist L and M such that for every $n > L$, $d(x_n, x) < \epsilon/2$ and for every $n > M$, $d(y_n, y) < \epsilon/2$. Let $N = \max\{L, M\}$. Then for every $n > N$, we have

$$|d(x_n, y_n) - d(x, y)| \leq d(y_n, y) + d(x_n, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad \blacksquare$$

Finally, we show that if (x_n) is a convergent sequence in (X, d) that converges to x, then every subsequence $(x_{n_k})_{k=1}^{\infty}$ converges to x as well.

Proof. Let (x_{n_k}) be a subsequence where n_k is a strictly increasing sequence of natural numbers. That is, $n_k < n_{k+1}$ for all k. Since $n_1 \geq 1$, it follows that $n_k \geq k$.

Let $\epsilon > 0$ be given. Then there exists N such that for all $n > N$, $d(x_n, x) < \epsilon$. Then for all $n > N$, $n_k > N$, hence for every $k > N$, $d(x_{n_k}, x) < \epsilon$. $\blacksquare$

We are now ready to prove the main proposition. Let

$$\delta := \inf\{d(a, b) : a \in A, b \in B\}.$$

Note that the set under "inf" is bounded below by 0 (by Def. 3.2(a)) and so δ exists (by the least upper bound property, Thm. 1.16), and $\delta \geq 0$ (by definition of the infimum).

There exists a sequence $((a_n, b_n)) \in A \times B$ such that

$$d(a_n, b_n) \to \delta$$

(by Ex. 3.11). Because A and B are compact there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ that converges to $a_0 \in A$ (by sequential compactness, Thm. 3.16). Then

$$d(a_{n_k}, b_{n_k}) \to \delta$$

as $k \to \infty$ by one of the lemmas proven in #2 (i.e. convergence $\Rightarrow$ convergence of every subsequence). Because $(b_{n_k})_{k=1}^{\infty}$ is itself a sequence in B (which is compact), there exists a subsequence $(b_{n_{k_s}})_{s=1}^{\infty}$ that converges to some $b_0 \in B$ (again by sequential compactness, Thm. 3.16). At the same time, the subsequence $(a_{n_{k_s}})_{s=1}^{\infty}$ still converges to the limit a_0 (by the same lemmas proven in #2). Hence

$$d(a_{n_{k_s}}, b_{n_{k_s}}) \to \delta$$

as $s \to \infty$, and by the above lemma, $d(a_0, b_0) = \delta$, which proves the claim.

(By the way, note that $\delta \neq 0$, as then (by Def. 3.2(a)) we must have $a_0 = b_0$, which contradicts the disjointness of A and B.)

#3 (a)

Let $\epsilon > 0$ be given. Because $(x_n) \to x$, there exists some N such that for all $n > N$, $|x_n - x| < \epsilon/2$. Then for all $n > N$,

$$\begin{aligned}\left|\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) - x\right| &= \left|\sum_{k=1}^{n} \frac{x_k - x}{n}\right| \\ &= \left|\sum_{k=1}^{N} \frac{x_k - x}{n} + \sum_{k=N+1}^{n} \frac{x_k - x}{n}\right| \\ &\le \left|\sum_{k=1}^{N} \frac{x_k - x}{n}\right| + \left|\sum_{k=N+1}^{n} \frac{x_k - x}{n}\right| \\ &\le \sum_{k=1}^{N} \frac{|x_k - x|}{n} + \sum_{k=N+1}^{n} \frac{|x_k - x|}{n} \\ &\le \frac{N}{n} \max_{1\le k\le N} \{|x_k - x|\} + \frac{\epsilon}{2}.\end{aligned}$$

Since $N \max_{1\le k\le N}\{|x_k - x|\}$ is a fixed quantity, there exists some $M > N$ such that for all $n > M$, $N \max_{1\le k\le N}\{|x_k - x|\} < \epsilon/2$. Thus for all $n > M$, we have

$$\left|\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) - x\right| < \epsilon.$$

#3 (b)

Let (x_n) be such that $x_k = 1$ if k is odd, and $x_k = -1$ if k is even. Then this sequence does not converge (as there are two subsequences converging to different limits), yet the average converges to 0 since $\sum_{k=1}^{n} x_k = 0$ or 1 for each n. Thus for every $\varepsilon > 0$

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k\right| \le \frac{1}{n} \le \varepsilon$$

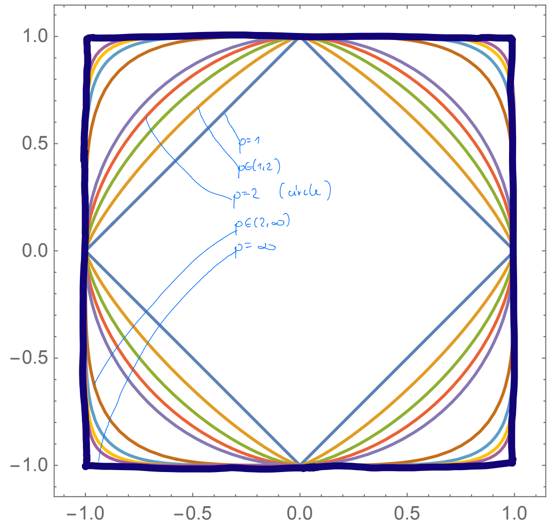
for $n \ge 1/\varepsilon$.

#4

No. Take $x = \pi$ and $y = 0$. Then $d(x, y) = |\sin \pi| = 0$, but $x \neq y$ (and so the nondegeneracy condition of a metric, Def. 3.2(a), is violated).

#5 (a)

For $p = 1$, $N_1(0)$ will correspond to the points (x_1, y_2) on $\mathbf{R}^2$ such that $|x_1| + |x_2| < 1$. This is the set of all points inside of the rhombus ("diamond") centered at the origin. As p increases, the boundary becomes more and more like a square of side length 2 centered at the origin (occurs when $p = \infty$). When $p = 2$, it is the unit circle.



#5 (b)

We will show 2 things: (i) For all $p < q$ and x, y, $d_p(x,y) \le d_q(x,y)$; and (ii) There exists a constant $c > 1$ independent of x, y so that $d_1(x,y) \le c d_\infty(x,y)$ for every x, y. This in turn proves that for every $p < q$,

$$d_p(x,y) \le c d_\infty(x,y) \le c d_q(x,y),$$

and

$$d_q(x,y) \le d_p(x,y) \le c d_p(x,y).$$

Observe that to show (i), it suffices to show that for $0 < t \le 1$, $(1+t^q)^{1/q} \le (1+t^p)^{1/p}$ whenever $p < q$. Indeed, this inequality is true if and only if $1 + t^q \le (1+t^p)^{q/p}$, and since $0 < t \le 1$, we have

$$(1+t^p)^{q/p} \ge 1 + t^p \ge 1 + t^q$$

because $q/p > 1$. Let $p < q$. Assume without loss that $|x_1 - y_1| \ge |x_2 - y_2|$. Then $t := |x_2 - y_2| / |x_1 - y_1| \le 1$, and so

$$\begin{aligned} d_q(x,y) &= (|x_1 - y_1|^q + |x_2 - y_2|^q)^{1/q} \\ &= |x_1 - y_1|(1+t^q)^{1/q} \\ &\le |x_1 - y_1|(1+t^p)^{1/p} \\ &= (|x_1 - y_1|^p + |x_2 - y_2|)^{1/p} = d_p(x,y). \end{aligned}$$

Observe that $d_1(x, y) \leq 2 \max\{|x_1 - y_1|, |x_2 - y_2|\} = 2d_\infty(x, y)$, hence by putting $c = 2$, the assertion is proved.

#6 (a)

Evidently $d(p_n, q_n) \geq 0$ for every p_n, q_n. Suppose $d(p_n, q_n) = 0$. Then we must have every $|a_k - b_k| = 0$, which is if and only if $a_k = b_k$, which is if and only if $p_n = q_n$.

Since d is evidently symmetric, we show that the triangle inequality holds. Let $p_n, q_n, r_n \in P_n$ be arbitrary. Here, $r_n = \sum_{k=0}^{n} c_k x^k$ with $c_k \neq 0$. Then

$$\begin{aligned} d(p_n, q_n) + d(q_n, r_n) &= \sum_{k=0}^{n} |a_k - b_k| + \sum_{k=0}^{n} |b_k - c_k| \\ &= \sum_{k=0}^{n} |a_k - b_k| + |b_k - c_k| \\ &\geq \sum_{k=0}^{n} |a_k - c_k| = d(p_n, r_n). \end{aligned}$$

Thus d is a metric by definition (Def. 3.2).

#6 (b)

$\tilde{d}$ is not a metric because $p_n(x) := x^n + 1$ and $q_n(x) := x^n - 1$ satisfy $\tilde{d}(p_n, q_n) = 0$ (if $k \neq 0$), yet they are distinct polynomials (so the nondegeneracy condition (Def. 3.2(a)) is violated). The case $k = 0$ is similar (take, for example, $p_n(x) := x^n$ and $q_n(x) := -x^n$).

#7

X is open because every neighborhood of every point is in X. Since $\varnothing = X^c$, $\varnothing$ is closed (by Thm 3.10). However, $\varnothing$ is open vacuously (it contains a neighborhood of any point in it, which there are none), hence X is also closed (by Thm 3.10).

#8

Let $\epsilon > 0$ be given. Then there is some N such that for all $n > N$, $|x_n - x| < \epsilon$. For all $n > N$, by one of the lemmas proved in #2 (or by PS2.7),

$$||x_n| - |x|| \leq |x_n - x| < \epsilon.$$

As for the second claim note that

$$\begin{aligned}
|(x_n - 1)^2 - (x-1)^2| &= |(x_n - 1 + x - 1)(x_n - 1 - x + 1)| \\
&= |x_n - x||(x_n - x) + 2(x-1)| \\
&\leq |x_n - x|^2 + 2|x-1|\,|x_n - x| \\
&\leq \epsilon
\end{aligned}$$

whenever N is sufficiently large so that $|x_n - x| \leq \min\{\sqrt{\epsilon/2}, \epsilon/(4|x-1|)\}$ (if $x \neq 1$; otherwise the term $2|x-1|\,|x_n - x|$ does not show up and the claim follows similarly).

#9

We want to show that if every sequence of K has a subsequence that converges in K, then K is closed. That is, every limit point of K is contained in K. Assume x is a limit point of K. Let (x_n) be a sequence in K that converges to x so that $x_n \neq x$ for every n. Then there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to an element in K by compactness (Thm. 3.16), and by one of the lemmas proven in #2 (i.e. convergence $\Rightarrow$ convergence of every subsequence), this limit is x. Thus $x \in K$, and so K is closed.

#10

Let $\{B_\alpha\}_{\alpha\in I}$ be any open cover of A. Let B_{α_0} be an element of this cover that includes 0 ($0 \in A$ so it must belong to at least one elements of the cover of A). Since B_{α_0} is open, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset B_{\alpha_0}$ (by Def. 3.8(c)). Since $1/n \to 0$ as $n \to \infty$ (by Ex. 3.6) we must have

$$1/n \in (-\varepsilon, \varepsilon) \subset B_{\alpha_0}$$

for $n \geq N$, where $N \in \mathbb{N}$ is sufficiently large (note N is given by the definition of convergence, Def. 3.4). So it remains to cover the remaining $N-1$ elements 1, 1/2, ... $1/(N-1)$ of A. Each of them belongs to at least one element of the cover $\{B_\alpha\}_{\alpha\in I}$, so altogether we have extracted N elements of this cover, that still cover A. Therefore A is compact by definition (Def. 3.14).

(One could alternatively solve this exercise by using sequential compactness (Thm. 3.16) and noting that any subsequence of $(1/n)_{n\geq 1}$ converges to 0.)